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Recent Results on Mandelbrot Multiplicative Cascades

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Abstract. This article gives a brief account on twenty five years of research on Mandelbrot multiplicative cascades with a stress on recent results on their multifractal analysis.

1. Introduction

Twenty five years ago, Mandelbrot [41–43] defined multiplicative processes which converge as random measures. This definition comes from his discussion of several works on turbulence [34, 56, 64] and in particular his refutation of the hypothesis of log-normality of spatial means of the dissipation of energy in a turbulent fluid. These processes model turbulence as well as intermittency, for instance phenomena such as the repartition of rare minerals in the earth crust. Random coverings [40] are also closely related to these cascades.

The best reference to have an overview of what was known on this topic at the beginning of the eigthies is Mandelbrot's book [44], "The Fractal Geometry of Nature". Since then and till nowadays, these processes have been widely investigated. This article aims at giving an overview of twenty five years of research on this topic and to provide a guide for reading this important literature.

2. The Mandelbrot Martingales

The b-adic tree

An integer $b \geq 2$ is given. Consider the set A^* of words on the alphabet $A = \{0, 1, 2, \dots, b-1\}$: $A^* = \{\epsilon\} \cup \bigcup_{n \geq 1} A^n$ (where ϵ is the empty word).

Endowed with the operation of concatenation, which will be denoted by a dot, A^* is a monoid with ϵ as unit element.

A^* is also endowed with a rooted tree structure: the root is ϵ, and, if $a \in A^*$, the vertices immediately following a are the $a.j$, for $0 \leq j < b$.

The following notations will prove convenient. If $a \in A^*$, its length will be denoted by $|a|$ and its j-th element by a_j; if $m \leq |a|$, $a_{|m}$ will stand for the word of length m whose letters are the m first ones of a.

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The boundary of this tree is the set $X = A^{\mathbb{N}^*}$ of sequences of elements of A. The set $A^* \cup X$ is an ultrametric space is the usual way: if u and v are elements of $X = A^{\mathbb{N}^*}$, and if n is the length of their longest common prefix, then their distance is $d(u,v) = b^{-n}$. If $x \in X$, one defines $x_{|n}$ to be the prefix of length n of x. It is clear that $x_{|n}$ tends towards x as n goes to infinity. It will also prove convenient to denote also by a the set of points x in X such that $x_{|n} = a$. So, we identify the elements of A^* with the balls of the ultrametric space X.

The Mandelbrot martingales

We are given a random variable $W = (W_0, W_1, \dots, W_{b-1})$ which takes its values in $[0, +\infty[^b$. It is assumed that $\mathrm{E}\,(W_0 + W_1 + \cdots + W_{b-1}) = 1$. We consider a sequence $\{W(a)\}$ of independent variables, identically distributed with W, and indexed by the nodes of the tree A^* (we shall write $W(\epsilon) = W$ occasionally).

For each $n \geq 1$, set

$$Y_n = \sum_{a \in A^n} \prod_{j=1}^{n} W_{a_j}(a_{|j-1})$$

and consider the σ-field $\mathcal{A}_n$ generated by the variables $\{W(a) \mid |a| < n\}$. It is clear that Y_n is a martingale adapted to the filtration $\{\mathcal{A}_n\}$. Therefore, it converges with probability 1 towards a *r.v.* Y. But, as we cannot say anything about the uniform integrability of this non-negative martingale, we can only say that one has $\mathrm{E}\, Y \leq 1$.

In a similar way, for any $a \in A^*$, the martingale

$$Y_n(a) = \sum_{b \in A^n} \prod_{j=1}^{n} W_{b_j}(a.b_{|j-1})$$

converges towards a *r.v.* $Y(a)$, with probability 1.

One has

$$Y = \sum_{j=0}^{n} W_j Y(j), \tag{1}$$

and, more generally,

$$Y(a) = \sum_{b \in A^n} Y(a.b) \prod_{j=1}^{n} W_{b_j}(a.b_{|(j-1)}).$$

This means that with probability 1 we have defined a measure μ on X:

$$\mu(a) = Y(a) \prod_{j=1}^{|a|} W_{a_j}(a_{|(j-1)}).$$

Of course, the total mass of μ is the *r.v.* Y.

Let us come back to equation (1). In it, the variables W and $Y(0), Y(1), \dots,$ and $Y(b-1)$ are independent, and the last ones equidistributed with Y.

In other terms, this construction provides a (hopefully non trivial) solution to the equation

$$Z \stackrel{\text{dist}}{=} \sum_{j=0}^{n} W_j Z(j), \tag{2}$$

where the variables appearing in this formula, obeys independence and distributions requirements analogous to those in (1). Indeed, the unknown is the distribution of Z rather than the *r.v.* Z itself.

3. Non degeneracy, moments and dimension

In this section, to avoid complications in the statements, we assume that the variable W is such that, with probability 1, all its component are nonzero.

Theorem 1. *The following assertions are equivalent:*

- $\mathrm{E}(Y) = 1$,
- *equation (2) has a solution with nonzero expectation,*
- $\sum_{j=0}^{b-1} \mathrm{E}(W_j \log W_j) < 0$.

Theorem 2. *For $h > 1$, the following assertions are equivalent:*

- $0 < \mathrm{E}(Y^h) < +\infty$,
- $\sum_{j=0}^{b-1} \mathrm{E}(W_j^h) < 1$.

Theorem 3. *If, for an $h > 1$ one has $\sum_{j=0}^{b-1} \mathrm{E}(W_j^h) < 1$. then*

$$\lim_{n\to\infty} \frac{\log \mu(x|_n)}{\log b^{-n}} = -\sum_{j=0}^{b-1} \mathrm{E}(W_j \log_b W_j)$$

with probability 1, for μ-almost every x.

Theorem 4. *If, for one $h > 0$, one has $\sum_{j=0}^{b-1} \mathrm{E}(W_j^{-h}) < \infty$, then, for any h' with $0 < h' < h$, one has $\mathrm{E}(Y^{-h'}) < \infty$.*

When the components of W are independent, theorems 1 and 2 are due to Kahane [31, 33], and theorem 3 to Peyrière [59, 33] and solve conjectures by Mandelbrot [43]. The proofs extend readily to the case considered here. The statement of theorem 4 is far from being the best possible (see [32, 55, 6]). Durrett and Liggett [20] gave a completely different proof of theorem 1, and their proof generated further studies, mainly by Liu [36–38], but also by Barral [7]. Guivarc'h [24] studied the limiting case $h > 1$ and $\sum_{j=0}^{b-1} \mathrm{E}(W_j^h) = 1$ in theorem 2. There are at least two non trivial generalisations of these cascades (but they imply the use of heavier a formalism): the case of random b, studied by Peyrière [60] in a particular case, and by Liu [37] in general; the case where nodes also are endowed with colour has been studied by Ben Nasr [9].

We are going to give a few hints on the proofs of some of these results.

Theorem 2

One has $Y = \sum_{j=0}^{b-1} W_j Y(j)$, from which it follows that $Y^h \geq \sum_{j=0}^{b-1} W_j^h Y(j)^h$, this inequality being strict with positive probability if Y is not equal to 0 with probability 1. Therefore, due to properties of independence and distribution, one has $1 > \sum_{j=0}^{b-1} \mathrm{E}(W_j^h)$ when $\mathrm{E}(Y^h) < \infty$.

The converse is more difficult to be obtained in full generality. We shall only consider the case $h = 2$. Since

$$\mathrm{E}(Y_n^2) = \mathrm{E}(\sum_{j=0}^{b-1} W_j Y_{n-1}(j))^2 = \mathrm{E}(Y_{n-1}^2)\ \mathrm{E}(\sum_{j=0}^{b-1} W_j^2) + 2\ \mathrm{E}(\sum_{0 \leq i < j < b} W_i W_j),$$

the martingale Y_n is bounded in L^2 when $\sum_{j=0}^{b-1} \mathrm{E}(W_j^2) < 1$.

Theorem 3

We may consider the random measure μ as a transition kernel and define a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{A}) \times (X, \mathcal{B})$, where $(\Omega, \mathcal{A})$ is the underlying probability space and $\mathcal{B}$ is the σ-field of Borel subsets of X:

$$\mathbb{P}(F) = \mathrm{E} \int_X \chi_F \, d\mu,$$

where χ_F is the indicator function of F.

The interest of considering this probability is that $\mathbb{P}$-almost surely means with probability 1, μ-almost everywhere. The corresponding expectation will be denoted by $\mathbb{E}$.

If $x \in X$, one has

$$\frac{1}{n} \log \mu(x_{|n}) = \frac{\log W_{x_1} + \log W_{x_2}(x_1) + \cdots + \log W_{x_n}(x_{|n-1})}{n} + \frac{1}{n} \log Y(x_{|n}).$$

The first term tends towards $\sum_{j=0}^{b-1} \mathrm{E}(W_j \log W_j)$ with $\mathbb{P}$ probability 1 (just use a martingale argument).

We wish to show that the second term goes to 0. To do so, consider the random function $T_n(x) = Y(x_{|n})$ $(x \in X$, and $n \in \mathbb{N})$. We have $\mathbb{E}(T_n^\alpha) = \mathrm{E}(Y^{1+\alpha})$; this is finite is $|\alpha|$ is small enough. The conclusion follows from the Borel-Cantelli lemma: for a fixed small positive α, with $\mathbb{P}$ probability 1, eventually one has $n^{-2/\alpha} \leq T_n \leq n^{2/\alpha}$.

Theorem 4

To avoid technicalities, we suppose moreover that the components of W are *i.i.d.*. If F_Y stands for the generating function of Y, *i.e.* $F_Y(t) = \mathrm{E}\ \mathrm{e}^{-tY}$, equation (2) gives

$$F_Y(t) = \left[\int_0^{+\infty} F(tu) \, dP_{W_0}(u) \right]^b,$$

where P_{W_0} is the distribution of W_0. Then it is not difficult to get $F_Y(t) = \mathrm{O}(t^{-h'})$ for some h' under the assumption $\mathrm{E}(W_0^{-h}) < \infty$. Of course the point in [32, 55, 6] is to obtain the right order of magnitude of the generating function and to get rid of the here assumed independence and non vanishing properties of components of W.

4. Simultaneous cascades

In this section, we are given a sequence $\{(W(a), W'(a))\}_{a\in A^*}$ of *i.i.d. r.v.*'s assuming their values in $\mathbb{R}^b \times \mathbb{R}^b$. We suppose that both variables W and W' give birth to non-degenerate Mandelbrot martingales Y_n and Y'_n whose limits are Y and Y'. The corresponding random measures on X will be denoted by μ and μ'.

Proposition 1. *Assume that, for one* $h \in]0,1[$, $\mathrm{E}\sum W_j^{1+h} < 1$, $\mathrm{E}\sum W_j'^{1+h} < 1$, *and* $\mathrm{E}\sum W_j'^{-h} < \infty$. *Then, one has*

$$\lim_{n\to\infty} \frac{\log \mu'(x|_n)}{\log b^{-n}} = \sum_{j=0}^{b-1} \mathrm{E}(W \log_b W'),$$

with probability 1, for μ-almost all x.

Proof. The first part of the proof of theorem 3 is unchanged. If ones keeps the notations of the sketch of proof of theorem 3, one has $\mathbb{E}(T_n'^{\gamma}) = \mathrm{E}(Y'^{\gamma} Y)$, which, due to Hölder's inequality and theorems 2 and 4, is finite when $|\gamma|$ is small enough. As previously, the conclusion comes from the Borel-Cantelli lemma.

Theorem 5. *Let* $h \in]1,2]$, *and suppose that* $\mathrm{E}\, Y^h$ *and* $\mathrm{E}\, Y'^h$ *are finite. Then there exists a constant C, which depends only on h and (W, W'), such that*

$$\mathrm{E}\,|Y - Y'|^h \le C \sum_{j=0}^{b-1} \mathrm{E}\,|W_j - W'_j|^h.$$

Proof. We use equation (2):

$$Y - Y' = \sum_{j=0}^{b-1} (W_j - W'_j)\, Y(j) + \sum_{j=0}^{b-1} W'_j (Y(j) - Y'(j)).$$

It results from an improved version of a lemma by Esseen and von Bahr [5] that

$$\mathrm{E}\left|\sum_{j=0}^{b-1} W'_j \left(Y(j) - Y'(j)\right)\right|^h \le 2\,\mathrm{E}\,|Y - Y'|^h \sum_{j=0}^{b-1} \mathrm{E}\, W_j'^h.$$

If $2\sum_{j=0}^{b-1} \mathrm{E}\, W_j'^h < 1$, we are done. Otherwise, one has to use equation (2) sufficiently many times (see [6] for details).

At this stage, we need to change slightly the notations: W' becomes L and μ and μ' will be denoted by μ_W and μ_L respectively.

We consider the lexicographic order $\prec$ on X and consider the random mapping γ from X to $\mathbb{R}$ so defined:

$$\gamma(x) = \mu_L\left(\{y \in X \mid y \prec x\}\right)$$

If $a \in A^*$, then the image of a, viewed as a ball in X, under γ is a random interval $I(a)$, which is semi-open to the right and whose length $|I(a)|$ equals $\mu_L(a)$. We consider the measure μ, image of μ_W under γ. In other terms, the μ-measure of the random interval $I(a)$ is $\mu_W(a)$.

This is the measure which we consider in the next sections.

5. Multifractal analysis

A multifractal formalism

We give first the version of the multifractal formalism which we need.

We are given semi-open sub-intervals $\{I(a)\}_{a\in A^*}$ of an interval I of $\mathbb{R}$, in such a way that $\left\{\{I(a)\}_{a\in A^n}\right\}_{n\geq 1}$ be a sequence of nested partitions. Let μ be a positive Borel measure on $\overline{I}$. For $q \in \mathbb{R}$, we set

$$C_n(q,t) = \sum_{a\in A^n}^{*} \mu\big(I(a)\big)^q |I(a)|^t$$

and

$$C(q,t) = \limsup_{n\to\infty} C_n(q,t),$$

where the star means that the summation runs only on a's such that $\mu(a) \neq 0$.

These are log-convex functions. Therefore, there exists a convex and non-increasing function τ, with values in $\overline{\mathbb{R}}$ such that $C(q,t) = \infty$ if $t < \tau(q)$ and $C(q,t) = 0$ if $t > \tau(q)$.

Also we define

$$E_\alpha = \left\{x \in I \;\Big|\; \lim_{n\to\infty} \frac{\log\mu(I_n(x))}{\log|I_n(x)|} = \alpha\right\},$$

where $I_n(x)$ stands for the interval of the form $I(a)$ with $a \in A^n$, and consider the Legendre transform of τ:

$$\tau^*(\alpha) = \inf_{q\in\mathbb{R}}\big(\alpha q + \tau(q)\big).$$

Proposition 2. *One has* $\dim E_\alpha \leq \tau^*(\alpha)$, *where* dim *stands for the Hausdorff dimension.*

Let us outline a proof of this fact. Fix α and suppose there exists q such that $\tau^*(\alpha) = \alpha q + \tau(q)$ (if it does not the reader will provide the slight changes to be made). We only detail the case $q > 0$. Let η be a small positive number.

Set

$$G_n(\alpha,\varepsilon) = \left\{a \in A^n \;\Big|\; \left|\frac{\log\mu\big(I(a)\big)}{\log|I(a)|} - \alpha\right| \leq \varepsilon\right\}.$$

We have

$$\sum_{a\in G_n(\alpha,\varepsilon)} |I(a)|^{\tau^*(\alpha)+\eta} = \sum_{a\in G_n(\alpha,\varepsilon)} |I(a)|^{(\alpha+\varepsilon)q}|I(a)|^{\tau(q)+\eta-\varepsilon q}$$
$$\leq C_n\big(q,\tau(q)+\eta-\varepsilon q\big).$$

and

$$E_\alpha = \bigcap_{\varepsilon>0}\bigcup_{m\geq 1}\bigcap_{n\geq m}\bigcup_{a\in G_n(\alpha,\varepsilon)} I(a).$$

On the other hand, since $\lim_{n\to\infty} C_n(q,\tau(q)+\eta/2) = 0$, there exists an increasing sequence of integers $\{n_j\}$ such that the series $\sum_{j\geq 1} C_{n_j}(q,\tau(q)+\eta/2)$ converges. It follows that, for any $j_0 > 0$, one can construct coverings $\{I_k\}$ of E_α such that $\sum |I_k|^{\tau^*(\alpha)+\eta} \leq K\sum_{j>j_0} C_{n_j}(q,\tau(q)+\eta/2)$, where K is a suitable constant.

If the series $\sum_{n\geq 1} C_n(q,\tau(q)+\eta)$ converges for any $\eta > 0$, we can consider packing dimension instead of Hausdorff's.

We say that the measure μ satisfies the multifractal formalism at α if the equality $\dim E_\alpha = \tau^*(\alpha)$ holds.

One can make a few remarks. Let us introduce, according to [35], the following quantities

$$f_\varepsilon(\alpha) = \inf\left\{t\in\mathbb{R} \mid \lim_{n\to\infty}\sum_{a\in G_n(\alpha,\varepsilon)} |I(a)|^t = 0\right\},$$

and

$$f(\alpha) = \lim_{\varepsilon\searrow 0} f_\varepsilon(\alpha).$$

What we proved in fact is that $\dim E_\alpha \leq f(\alpha) \leq \tau^*(\alpha)$. The proof of $f\leq\tau^*$ is close to the proof of Chernoff inequality [16]. Fairly often, when the intervals of the same generation have the same length, the converse inequality comes from Cramér's theorem [18]. So, one may think of theorems of the multifractal formalism as results on large deviations.

The term multifractal appeared in [22], the generalized dimensions in [26], and the connection with the thermodynamical formalism in [25], but Chernoff's inequality is used in [43]. The first description of a behaviour, nowadays called multifractal, seems to appear while studying the errors in a communication channel [39].

The first formulations of this formalism [25, 26] dealt with a sequence of nested partitions (not necessarily b-adic) of the interval $[0,1]$. They gave rise to many studies, in mathematics and in other sciences as well [45–54], also in dynamical systems and geometric measure theory [17, 14, 62, 59, 10, 63]. A formalism "without boxes", and therefore more satisfactory from the point of view of geometric measure theory, has been defined by Olsen [57]. It has been investigated by Ben Nasr and Bhouri [11, 12] who gave new conditions for a measure to satisfy it.

The first applications were the analysis of turbulence. Several authors developed a multifractal analysis of signals more general than primitives of measures in connection with wavelets theory [23, 28, 2–4, 19, 29, 30].

Multifractal analysis of Mandelbrot measures

According to the above formalism, we wish to study the random quantity

$$C_n(q,t) = \sum_{a\in A^n} \mu_W(a)^q \mu_L(a)^t$$

One can check that

$$\mathrm{E}\, C_n(q,t) = \left[\sum_{j=0}^{b-1} \mathrm{E}\, W_j^q L_j^t\right]^n \mathrm{E}\, Y_W^q Y_L^t. \tag{3}$$

This leads to consider the following functions

$$\varphi(q,t) = \sum_{j=0}^{b-1} \mathrm{E}\, W_j^q L_j^t$$

and

$$\psi(q,t) = \mathrm{E}\, Y_W^q Y_L^t.$$

These are log-convex functions. We make the assumption that the points $(0,0)$, $(0,1)$, and $(1,0)$ are in the interior of the convex set where φ and ψ are finite.

The lower part of the boundary of the set $\{\varphi < 1\}$ is the graph of a convex function $\tilde{\tau}(q)$.

Theorem 6. *Under the previous assumptions,*

(1) *there exists an open interval $\mathcal{I}$ containing 0 and 1 on which the functions τ and $\tilde{\tau}$ agree;*
(3) *for any $q \in \mathcal{I}$, with probability 1 the measure μ satisfies the multifractal formalism at $-\tau'(q)$.*

Proof. If $\varphi(q,t) < 1$ and $\psi(q,t) < \infty$, it results from (2) that, with probability 1, $C_n(q,t)$ goes to 0 as n goes to ∞. This proves that, on a suitable interval, one has $\tau \le \tilde{\tau}$ with probability 1.

To prove the converse we shall consider a Gibbs measure. Fix q, and consider the multivariate *r.v.* W' so defined:

$$W'_j = W_j^q L_j^{\tilde{\tau}(q)} \Big/ \left[\sum_{k=0}^{b-1} \mathrm{E}\, W_k^q L_k^{\tilde{\tau}(q)}\right] = W_j^q L_j^{\tilde{\tau}(q)}$$

This variable gives rise to a measure $\mu_{W'}$ on X, and to a measure μ' on $\mathbb{R}$.

Due to proposition 1, with probability 1 for μ'-almost all x, one has

$$\lim_{n\to\infty} \frac{\log \mu'(I_n(x))}{\log |I_n(x)|} = \frac{\sum_{j=0}^{b-1} \mathrm{E}\big[W_j^q L_j^{\tilde{\tau}(q)} \log(W_j^q L_j^{\tilde{\tau}(q)})\big]}{\sum_{j=0}^{b-1} \mathrm{E}\big[W_j^q L_j^{\tilde{\tau}(q)} \log L_j\big]} = \tilde{\tau}(q) - q\tilde{\tau}'(q)$$

and

$$\lim_{n\to\infty} \frac{\log |I_n(x)|}{n} = \sum_{j=0}^{b-1} \mathrm{E}\big[W_j^q L_j^{\tilde{\tau}(q)} \log L_j\big].$$

This implies, by using a generalization [61] of a classical lemma by Billingsley [13], that any Borel set E such that $\mu'(E) > 0$ has a Hausdorff dimension larger than or equal to $\tilde{\tau}^*(-\tilde{\tau}'(q))$.

But, still due to proposition 1, one has, with probability 1 for μ'-almost every x,

$$\lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \frac{\sum_{j=0}^{b-1} \mathrm{E}\big[W_j^q L_j^{\tilde{\tau}(q)} \log W_j\big]}{\sum_{j=0}^{b-1} \mathrm{E}\big[W_j^q L_j^{\tilde{\tau}(q)} \log L_j\big]} = -\tilde{\tau}'(q),$$

which means that $E_{-\tilde{\tau}'(q)}$ has full μ' measure, and therefore has a Hausdorff dimension $\geq \tilde{\tau}^*(-\tilde{\tau}'(q))$. Therefore, we have proved the equality $\tilde{\tau}^*(-\tilde{\tau}'(q)) = \tau^*(-\tilde{\tau}'(q))$. But, as this result holds with probability 1 for countably many q's at the same time, we conclude that $\tau = \tilde{\tau}$ (at least on an open interval containing 0 and 1).

Remark. For each q we considered a measure $\mu_q = \mu'$. As a consequence of theorem 5, it is possible to perform a modification of the process $q \mapsto \mu_q$ so that its trajectories be continuous with probability 1. This means that, with probability 1, the Gibbs measures can be simultaneously defined. But this is not enough to prove the following stronger statement: with probability 1, for any $q \in \mathcal{I}$, the measure μ satisfies the multifractal formalism at $-\tau'(q)$. This result has just been proved by Barral [8].

Theorem 6, as stated here, is due to Barral [6]. Indeed his results are more general: in particular, they also deal with the case where components of W and L may vanish with positive probability. Molchan [56] obtained analogous results when dealing with one Mandelbrot martingale. Previous formulations existed: Kahane [32] obtained $\dim E_\alpha \geq \tilde{\tau}^*(\alpha)$, Holley and Waymire [27] got the equality under the hypothesis $\min_{0\leq j<b} \operatorname{ess\,inf} W_j > 0$. See also [1, 15, 21] for connected works.

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