On the dynamic behaviours of the iterations of the trace map associated with substitutive sequence *

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Abstract

Let σ be a substitution on a two-letter alphabet, let Φ_σ be the polynomial trace map associated with σ of which the existence has been proved by J.-P. Allouche and J. Peyrière, and let $\Omega = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 - xyz - 4 = 0\}$. We shall discuss in this paper the asymptotic properties of Φ_σ^n, the iterations of Φ_σ. The dynamic behaviours such as invarint set, fixed points, periodic orbits and chaotic properties are studied in detail. We show that there properties depond strongly on the difference parts of Ω and they are determined by the substitutive matrix M_σ of the substitution σ.

The discovery of quasicristals by D. Schechtman *et al.*[9] has given rise to many studies of ordered but no periodic system. Among these studies, one-dimensional systems generated by a substitution σ acting upon a two-letter alphabet $\mathcal{S} = \{a, b\}$ are particularly interesting and important, because in this case, there is a general theorem [1] yielding a recusion formula for the traces of certain products of transfer matrices. The properties of these trace maps are studies by many authurs [2, 3, 5, 4, 6, 7, 11]. In these studies, the surface $x^2 + y^2 + z^2 - xyz - 4 = 0$ play an impotant role, due to some trace formulas being satisfied over this surface. The aim of this paper is devoted to study the dynamic behaviours of the iterations of the trace map Φ_σ over Ω. This work is organized as follows: in section 1, some notations and definitions are introduced and some elementary results are

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established. In section 2, we study in detail the dynamic behaviours of Φ_σ^n, the iterations of the trace map Φ_σ, as we shall see, the properties, such as invariant set, periodic orbits, fixed point and chaotic behaviours etc. will depend strongly on the difference domains of Ω and they will be determined by the substitutive matrix M_σ of the substitution σ. Some examples are illustrated in section 3.

1 Preliminaries

In this paper, we adopt the following notations.

1. Let $\mathcal{S} = \{a, b\}$ be an alphabet of two letters, we note $\mathcal{S}^*$ the monoid of words over $\mathcal{S}$ and F the free group generated by $\mathcal{S}$. A substitution σ on $\mathcal{S}$ is a homomorphism of $\mathcal{S}^*$ into itself, which is a particular endomorphism of F.

If σ and τ are elements of Hom(F,F), we set $\sigma\tau = \tau \circ \sigma$, where Hom$(F,F)$ denotes the set of homomorphism from F to F, and 'o' denotes the composition of functions.

2. $SL_2(\mathbb{C})$ denotes the set of 2×2 matrices with determinant ± 1 and entries of which are in $\mathbb{C}$. An element of Hom$(F, SL_2(\mathbb{C}))$ is uniquely determined by the couple $(\phi(a), \phi(b))$ of elements of $SL_2(\mathbb{C})$.

3. We define a map T: Hom$(F, SL_2) \to \mathbb{C}^3$ by the following manner:

$$T(\phi) = (\mathrm{tr}\,\phi(a), \mathrm{tr}\,\phi(b), \mathrm{tr}\,\phi(ab)),$$

where tr stands for the trace.

4. Define $\lambda(x,y,z) = x^2 + y^2 + z^2 - xyz - 4$, $\Omega = \{(x,y,z) \in \mathbb{R}^3;\ \lambda(x,y,z) = 0\}$, $\Omega^c = \mathbb{R}^3 \setminus \Omega$.

With these notations, J.-P. Allouche and J. Peyrière [1, 6] have proved the following result:

Theorem 1.1 *For any $\sigma \in Hom(F,F)$, there exists an unique $\Phi_\sigma \in (\mathbb{Z}[x,y,z])^3$ such that for any $\phi \in Hom(F, SL_2(\mathbb{C}))$, we have*

$$T(\phi \circ \sigma) = \Phi_\sigma(T(\phi)),$$

where $\mathbb{Z}[x,y,z]$ denotes the set of polynomials in the varibles x, y and z, the coefficients of which are integers.

We call Φ_σ the trace map associated with σ.

A direct consequence of Theorem above is

Corollary 1.2 *For σ and τ in $Hom(F,F)$, we have $\Phi_{\sigma\tau} = \Phi_\sigma \circ \Phi_\tau$. In paticular, $\Phi_\sigma^n = \Phi_\sigma \circ \Phi_\sigma \circ \cdots \circ \Phi_\sigma = \Phi_{\sigma}^{n}$.*

The following theorem [6],[11] describes some divisibility properties of the involved polynomials which play an important role in the studies of dynamic of Φ_σ^n.

Theorem 1.3 *For any $\sigma \in Hom(F,F)$, there existts a polynomial $Q_\sigma \in \mathbb{Z}[x,y,z]$, such that $\lambda \circ \Phi_\sigma = \lambda \cdot Q_\sigma$. Moreover, $Q_\sigma = 1$ if and only if $\sigma \in Aut\, F$, where $Aut\, F$ denotes the set of automorphism of F into itself.*

From Theorem 1.3, we obtain immediately:

Corollary 1.4 *For any $\sigma \in Hom(F,F)$, we have $\Phi_\sigma(\Omega) \subset \Omega$. In particular, if $\sigma \in Aut\, F$, then $\Phi_\sigma(\Omega_r) \subset \Omega_r$, where $r \in \mathbb{R}$, and $\Omega_r = \{(x,y,z) \in \mathbb{R}^3;\ \lambda(x,y,z) - r = 0\}$.*

The results stated above are still valid if we replace F by $\mathcal{S}^*$. In the most part of this paper, we shall confine our discussions over $\mathcal{S}^*$, so we introduce some notations.

Let $w \in \mathcal{S}^*$ be a word, we denote by $|w|_a$ (res. $|w|_b$) the number of 'a' (res. b) occuring in w, and we note $L(w)$ the vector $(|w|_a, |w|_b)^t$, where v^t is the transposition of the vector v.

Let σ be a substitution over S, its substitutive matrix M_σ defined by

$$M_\sigma = \begin{pmatrix} |\sigma(a)|_a & |\sigma(b)|_a \\ |\sigma(a)|_b & |\sigma(b)|_b \end{pmatrix} =: \begin{pmatrix} m_{aa} & m_{ba} \\ m_{ab} & m_{bb} \end{pmatrix}.$$

It is readily checked that for any $w \in \mathcal{S}^*$,

$$M_{\sigma^n} = M_\sigma^n = \begin{pmatrix} |\sigma(a)|_a & |\sigma(b)|_a \\ |\sigma(a)|_b & |\sigma(b)|_b \end{pmatrix}^n =: \begin{pmatrix} N_{n,a,a} & N_{n,b,a} \\ N_{n,a,b} & N_{n,b,b} \end{pmatrix}$$

and

$$L(\sigma^n(w)) = M_\sigma^n L(w). \tag{1}$$

Let Φ_σ be the trace map associated with σ, we write

$$\Phi_\sigma^n =: (\Phi_{\sigma,a}^n, \Phi_{\sigma,b}^n, \Phi_{\sigma,ab}^n)$$

From Theorem 1.1 and Corollary 1.2, if $A, B \in SL_2(\mathbb{R})$, and if we set $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, $z = \operatorname{tr} AB$, then

$$\Phi_{\sigma,s}^n(x,y,z) = \operatorname{tr} \phi(\sigma^n(s)), \quad s \in \{a, b, ab\} \tag{2}$$

Furthermore, if $(x, y, z) \in \Omega$, then by [6], the matrices A, B may be simultaneously triangulized. Thus A, B may be rewritten as the following form:

$$A = \begin{pmatrix} \mu & \mu^* \\ 0 & \mu^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \nu & \nu^* \\ 0 & \nu^{-1} \end{pmatrix} \tag{3}$$

where $\mu, \mu^*, \nu, \nu^* \in \mathbb{C}$.

Thus by (2) and (3), if we set $M_\sigma = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then for $(x, y, z) \in \Omega$, we have

$$\Phi_{\sigma,a} = \mu^p \nu^q + \mu^{-p} \nu^{-q}, \quad \Phi_{\sigma,b} = \mu^r \nu^s + \mu^{-r} \nu^{-s} \tag{4}$$

and

$$\begin{aligned} \Phi^n_{\sigma,a} &= \mu^{N_{n,a,a,}} \nu^{N_{n,a,b}} + \mu^{-N_{n,a,a}} \nu^{-N_{n,a,b}}, \\ \Phi^n_{\sigma,b} &= \mu^{N_{n,b,a,}} \nu^{N_{n,b,b}} + \mu^{-N_{n,b,a}} \nu^{-N_{n,b,b}} \end{aligned} \tag{5}$$

Remark 1.5 We do not write the formula for $\Phi^n_{\sigma,ab}$, Since for $(x, y, z) \in \Omega$, by Corollary 1.2, $(\Phi^n_{\sigma,a}, \Phi^n_{\sigma,b}, \Phi^n_{\sigma,ab}) \in \Omega$, hence $\Phi^n_{\sigma,ab}$ will be determined by $\Phi^n_{\sigma,a}$ and $\Phi^n_{\sigma,b}$.

Proposition 1.6 *Let σ, τ be two substitutions over $\mathcal{S}$, then $\Phi_\sigma|_\Omega = \Phi_\tau|_\Omega$ if and only if $M_\sigma = M_\tau$.*

PROOF. The part "if" follows immediately from (4).

To prove the part "only if", let $M_\tau = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$, then by (4)

$$\mu^p \nu^q + \mu^{-p} \nu^{-q} = \mu^{p'} \nu^{q'} + \mu^{-p'} \nu^{-q'}$$

holds for any $\mu, \nu \in \mathbb{C} \setminus \{0\}$. Taking $\mu = 1$, we get $q = q'$, and taking $\nu = 1$, we get $p = p'$. In the same way, we have also $r = r'$ and $s = s'$.

By means of Proposition 1.6, we see that the properties of Φ^n_σ over Ω will be completely determined by its substitutive matrix. In other words, substitutive marix will play an essential role in the studies of the trace map on Ω.

Now we devide Ω into three disjoint parts as follows:

$$\begin{aligned} \Omega_S &= \{(x, y, z) \in \Omega; \ |x| = |y| = 2\} \\ \Omega_0 &= \{(x, y, z) \in \Omega \setminus \Omega_S; \ |x|, |y| \geq 2\} \\ \Omega_i &= \{(x, y, z) \in \Omega \setminus \Omega_S; \ |x|, |y| \leq 2\} \end{aligned}$$

Notice that if $(x, y, z) \in \Omega$, then $z = \dfrac{xy \pm \sqrt{(x^2 - 4)(y^2 - 4)}}{2}$, so $|z|$ is greater than 2, equal to 2, and less than 2 respectively over Ω_0, Ω_S and Ω_i.

For convenience, we shall adopt some parametrizations of Ω_0 and Ω_i in section 2.

2 Dynamic behaviours of Φ_σ over Ω

We keep the notations and definitions introduced in section 1.

2.1 Ω_S

Ω_S consists of the four points $p_{00} = (2,2,2)$, $p_{01} = (2,-2,-2)$, $p_{10} = (-2,2,-2)$, $p_{11} = (-2,-2,2)$, which are exactly the singular points of the surface Ω.

We obtain easily from (4) the following formular:

$$\Phi_\sigma(p_{\epsilon\eta}) = ((-1)^{\epsilon p+\eta q}2, (-1)^{\epsilon r+\eta s}2, (-1)^{\epsilon(p+r)+\eta(q+s)}2), \tag{6}$$

where $\epsilon, \eta \in \{0,1\}$.

The formula (6) shows that Φ_σ is determined by the parity of the elements of the substitutive matrix M_σ for $(x,y,z) \in \Omega_S$. Therefore if we set $\overline{M_\sigma} = \begin{pmatrix} \overline{m_{aa}} & \overline{m_{ba}} \\ \overline{m_{ab}} & \overline{m_{bb}} \end{pmatrix}$, where $\overline{m_{st}} \in \{0,1\}$ and $\overline{m_{st}} \equiv m_{st} \pmod 2$, $s,t \in \{a,b\}$, then the values of Φ_σ over Ω_S will be determined by $\overline{M_\sigma}$. It is readily checked that there are 16 possible types of such matrices as follows:

$$\overline{M_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{M_2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \overline{M_3} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overline{M_4} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\overline{M_5} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \overline{M_6} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \overline{M_7} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{M_8} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\overline{M_9} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{M_{10}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \overline{M_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{M_{12}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\overline{M_{13}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overline{M_{14}} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \overline{M_{15}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \overline{M_{16}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

From (6), we obtain the following table which sumarizes the dynamic behaviours of Φ_σ on Ω_S (fixed point, periodic orbit, attractive point):

	$\overline{M_1}$	$\overline{M_2}$	$\overline{M_3}$	$\overline{M_4}$	$\overline{M_5}$	$\overline{M_6}$	$\overline{M_7}$	$\overline{M_8}$	$\overline{M_9}$	$\overline{M_{10}}$	$\overline{M_{11}}$	$\overline{M_{12}}$	$\overline{M_{13}}$	$\overline{M_{14}}$	$\overline{M_{15}}$	$\overline{M_{16}}$
p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}	p_{00}
p_{01}	p_{01}	p_{11}	p_{10}	p_{01}	p_{11}	p_{10}	p_{00}	p_{10}	p_{00}	p_{11}	p_{00}	p_{01}	p_{00}	p_{11}	p_{10}	p_{01}
p_{10}	p_{10}	p_{01}	p_{11}	p_{11}	p_{10}	p_{01}	p_{00}	p_{00}	p_{01}	p_{11}	p_{10}	p_{00}	p_{11}	p_{00}	p_{10}	p_{01}
p_{11}	p_{11}	p_{01}	p_{01}	p_{10}	p_{01}	p_{11}	p_{00}	p_{10}	p_{01}	p_{00}	p_{10}	p_{01}	p_{11}	p_{11}	p_{00}	p_{00}

Table 1

2.2 Ω_0

Ω_0 can be divided further into disjoint connective domains $\Omega_0^{\epsilon\eta}$ as follows:

$$\begin{aligned}\Omega_0^{00} &= \{(x,y,z)\in\Omega_0;\ x\geq 2,\ y\geq 2\};\\ \Omega_0^{\epsilon\eta} &= \{((-1)^{\epsilon}x,(-1)^{\eta}y,(-1)^{\epsilon\eta}z);\ (x,y,z)\in\Omega_0^{00}\},\end{aligned}$$

where $\epsilon,\eta\in\{0,1\}$.

Ω_0^{00} has the following parameter reperesentation:

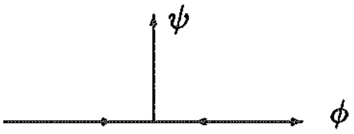

Figure 1

$$\left\{\begin{aligned} x &= 2\cosh\phi\\ y &= 2\cosh\psi\\ z &= 2\cosh(\phi\pm\psi)\end{aligned}\right.$$

Using these parametrizations, we can rewrite (5) as

$$\begin{aligned}\Phi^n_{\sigma,a}(x,y,z) &= (-1)^{\epsilon N_{n,a,a}+\eta N_{n,a,b}}2\cosh(N_{n,a,a}\phi+N_{n,a,b}\psi)\\ \Phi^n_{\sigma,b}(x,y,z) &= (-1)^{\epsilon N_{n,b,a}+\eta N_{n,b,b}}2\cosh(N_{n,b,a}\phi+N_{n,b,b}\psi)\end{aligned}\tag{7}$$

where $(x,y,z)\in\Omega_0^{\epsilon\eta}$.

Proposition 2.1 *Let* $\begin{pmatrix}\epsilon'\\ \eta'\end{pmatrix}\equiv\overline{M_\sigma}\begin{pmatrix}\epsilon\\ \eta\end{pmatrix}$ (mod 2), *then* $\Phi_\sigma(\Omega_0^{\epsilon\eta})\subset\Omega_0^{\epsilon'\eta'}$.

In fact, let $(x,y,z)\in\Omega_0^{\epsilon\eta}$, then by taking $n=1$ in (7), we have

$$\begin{aligned}\Phi^n_{\sigma,a}(x,y,z) &= (-1)^{p\epsilon+q\eta}2\cosh(p\phi+q\psi)=(-1)^{\epsilon'}2\cosh(p\phi+q\psi)\\ \Phi^n_{\sigma,b}(x,y,z) &= (-1)^{r\epsilon+s\eta}2\cosh(r\phi+s\psi)=(-1)^{\eta'}2\cosh(r\phi+s\psi)\end{aligned}$$

which yields the conclution.

We obtain readily from Proposition 2.1 the following corollary:

Corollary 2.2 *With the notations above, we have* $\Phi_\sigma(\Omega_0^{\epsilon\eta})\subset\Omega_0^{\epsilon_n\eta_n}$, *where* $\begin{pmatrix}\epsilon_n\\ \eta_n\end{pmatrix}\equiv\overline{M_\sigma}^n\begin{pmatrix}\epsilon\\ \eta\end{pmatrix}$ (mod 2).

Remark 2.3 We see that from (7), for any $\epsilon,\eta\in\{0,1\}$, and for any $(x,y,z)\in\Omega_0^{00}$, we have

$$|\Phi_{\sigma,i}((-1)^{\epsilon}x,(-1)^{\eta}y,(-1)^{\epsilon+\eta}z)|=|\Phi_{\sigma,i}(x,y,z)|,\quad i\in\{a,b,ab\}$$

On the other hand, by Corollary 2.2, the domain where Φ_σ^n visit is determined completely by $\overline{M_\sigma}$ hence by M_σ and by the initial point $(x,y,z) \in \Omega_0$. In fact, we can determine it according to Table 1. Thus in the following, we shall only discuss the dynamic behaviours of Φ_σ^n over Ω_0^{00}.

For discussing further the asymptotic properties of Φ_σ^n, we need the following facts which are essentially the consequences of the classic Perron-Fronenius Theorem (see [8] for example).

Lemma 2.4 *Let σ be a substitution over S and let M_σ be its substitutive matrix. Suppose that M_σ is primitive (i.e. there is an integer $N \in \mathbb{N}$, such that all entries of the matrix M_σ^N are strictly positive), and suppose that Δ and δ are the eigenvalues of M_σ with $|\Delta| \geq |\delta|$, then we have*

(1) $\Delta > |\delta|$;

(2) there exists positive numbers $d_a > 0$, $d_b > 0$, such that for any $w \in S^*$

$$\frac{L(\sigma_n(w))}{|\sigma^n(w)|} \longrightarrow \begin{pmatrix} d_a \\ d_b \end{pmatrix}, \quad n \longrightarrow \infty$$

and

$$\frac{N_{n,s,a}}{N_{n,s,b}} \longrightarrow \frac{d_a}{d_b}, \quad n \longrightarrow \infty, \quad s \in \{a,b\};$$

(3) Set $\alpha = \dfrac{d_a}{d_b} > 0$, then the vector $(1,-\alpha)$ is the left eigenvector of M_σ corresponding to the eigenvalue δ.

The Lemma 2.4(3) follows that

$$\begin{aligned} N_{n,a,a} - \alpha N_{n,a,b} &= (1,-\alpha)\begin{pmatrix} N_{n,a,a} \\ N_{n,b,a} \end{pmatrix} = (1,-\alpha)M_\sigma^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta^n(1,-\alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (8) \\ &= \delta^n, \\ N_{n,b,a} - \alpha N_{n,b,b} &= (1,-\alpha)M_\sigma^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\alpha\delta^n \qquad (9) \end{aligned}$$

Now let

$$S_\alpha = \{(x,y,z) \in \Omega_0^{00};\ \phi/\psi = -\alpha\}$$

where $x = 2\cosh\phi$, $y = 2\cosh\psi$ and α is defined in Lemma 2.4. Then by (7), (8) and (9)

$$\begin{aligned} \Phi_{\sigma,a}^n(x,y,z) &= 2\cosh(N_{n,a,a}\phi + N_{n,a,b}\psi) = 2\cosh(N_{n,a,a} - \alpha N_{n,a,b})\phi \\ &= 2\cosh(\delta^n\phi) \end{aligned} \qquad (10)$$

and

$$\Phi^n_{\sigma,b}(x,y,z) = 2\cosh(-\alpha\delta^n\phi). \tag{11}$$

Hence by (10) and (11) we obtain

Proposition 2.5 *With notations above, we have* $\Phi_\sigma(S_\alpha) \subset S_\alpha$.

Now we can state our main result of this subsection:

Theorem 2.6 *With notations above, we have*

(1) if $(x,y,z) \in \Omega_0^{00} \setminus S_\alpha$, *then* $\Phi^n_\sigma(x,y,z) \longrightarrow (\infty,\infty,\infty)$, $n \to \infty$;

(2) if $(x,y,z) \in S_\alpha$, *then*

$$\Phi^n_\sigma \begin{cases} \to & (\infty,\infty,\infty), \quad n \to \infty, \quad \textit{if } |\delta| > 1 \\ \to & (2,2,2), \quad n \to \infty, \quad \textit{if } |\delta| < 1 \\ = & (x,y,z), \quad n \geq 1, \quad \textit{if } |\delta| = 1 \end{cases}$$

PROOF. (1) Let $(x,y,z) \in \Omega_0^{00}$, then $\phi/\psi = \beta \neq -\alpha^{-1}$. Thus

$$N_{n,a,a} + N_{n,a,b}\psi = \phi N_{n,a,b}\left(\frac{N_{n,a,a}}{N_{n,a,b}} + \frac{\psi}{\phi}\right) = \phi N_{n,a,b}\left(\frac{N_{n,a,a}}{N_{n,a,b}} + \frac{1}{\beta}\right),$$

Since $\frac{N_{n,a,a}}{N_{n,a,b}} \to \alpha$ by Lemma 2.4(2), hence $\frac{N_{n,a,a}}{N_{n,a,b}} + \frac{1}{\beta}$ tends to a nozero number by $\beta \neq -\alpha^{-1}$. This $N_{n,a,a} + N_{n,a,b}\psi$ tends to infinity as n goes to infinity. Therefore $\Phi^n_{\sigma,a}(x,y,z) = 2\cosh(N_{n,a,a}\phi + N_{n,a,b}\psi)$ tends to infinity. The same analysis follows that $\Phi^n_{\sigma,b}$ and $\Phi^n_{\sigma,ab}$ also tend to infinity.

(2) Let $(x,y,z) \in S_\alpha$, then $\phi/\psi = \beta \neq -\alpha^{-1}$, so by (10) and (11)

$$\begin{aligned} \Phi^n_{\sigma,a}(x,y,z) &= 2\cosh(\delta^n\phi), \\ \Phi^n_{\sigma,b}(x,y,z) &= 2\cosh(-\alpha\delta^n\phi) \end{aligned}$$

(i) if $|\delta| > 1$, then $|\delta|^n \to \infty$, $n \to \infty$, thus $\Phi^n_\sigma(x,y,z) \to (x,y,z)$;

(ii) if $|\delta| < 1$, then $|\delta|^n \to 0$, $n \to \infty$, therefore $\Phi^n_\sigma(x,y,z) \to (2,2,2)$;

(iii) if $|\delta| = 1$, then

$$\begin{aligned} \Phi^n_{\sigma,a}(x,y,z) &= 2\cosh(\phi) = x, \\ \Phi^n_{\sigma,b}(x,y,z) &= 2\cosh(-\alpha\phi) = 2\cosh(\psi) = y, \\ \Phi^n_{\sigma,ab}(x,y,z) &= 2\cosh(\phi \pm (-\alpha\phi)) = 2\cosh(\phi \pm \psi) = z. \end{aligned}$$

Remark 2.7 From Theorem 2.6, we see that if $|\delta| > 1$, then all the points of Ω_0^{00} tend to infinity under Φ_σ^n, that is the unique "attractive" point of Φ_σ^n; if $|\delta| = 1$, then all the points of S_α are the fixed points of Φ_σ, furthermore, $\Phi_\sigma(S_\alpha) = S_\alpha$ and all the points outside of S_α tend to infinity, thus S_α is a repeller with respect to Φ_σ; finally, if $|\delta| < 1$, then the point $(2,2,2)$ is a "attractive" point, in this case, there two "attractiv"points, i.e. infinity and $(2,2,2)$. Thus, the dynamic behaviours of Φ_σ are completely determined by the little eigenvalue of M_σ.

Remark 2.8 If $\Omega_0^{\epsilon\eta} \neq \Omega_0^{00}$, then the asymptotic properties of Φ_σ^n are some more complex, but we can still completely determine these behaviours by Proposition 2.1, Corollary 2.2, Remark 2.3, Theorem 2.6 and Table 1.

Remark 2.9 If σ is an automorphism, then $\det M_\sigma = \pm 1$, see [7]. In this case, we have always $|\delta| < 1$.

Remark 2.10 If $\delta = 0$, then for $(x,y,z) \in S_\alpha$, $\Phi_\sigma(x,y,z) = (2,2,2)$; if $\delta \neq 0$, then S_α is invariant by Φ_σ, i.e. $\Phi_\sigma(S_\alpha) = S_\alpha$.

2.3 Ω_i

The dynamic behaviours over Ω_i are much more complex than that over Ω_0, which exihibit very strong chaotic properties.

The following parametrization of Ω_i will be convenient, see Figure 2.

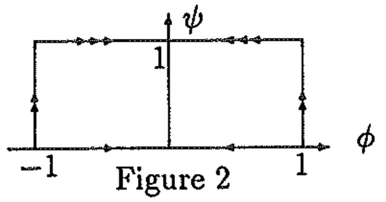

Figure 2

$$\left\{\begin{array}{rcl} x & = & 2\cos\phi\pi \\ y & = & 2\cos\psi\pi \\ z & = & 2\cos(\phi\pm\psi)\pi \end{array}\right.$$

Discuss as in section 2.2, we have

$$\Phi_{\sigma,a}^n = 2\cos(N_{n,a,a}\phi + N_{n,a,b}\psi)\pi \tag{12}$$

$$\Phi_{\sigma,b}^n = 2\cos(N_{n,b,a}\phi + N_{n,b,b}\psi)\pi \tag{13}$$

Let $\tilde{S}_\alpha = \{(x,y,z) \in \Omega_i;\ \phi/\psi = -\alpha^{-1}\}$, then as in section 2.2, we have

Proposition 2.11 $\Phi_\sigma(\tilde{S}_\alpha) \subset \tilde{S}_\alpha$. *In particular, if* $\delta \neq 0$, *then* $\Phi_\sigma(\tilde{S}_\alpha) = \tilde{S}_\alpha$.

Notice that for $(x,y,z) \in \tilde{S}_\alpha$, we have

$$\begin{aligned} N_{n,a,a}\phi + N_{n,a,b}\psi &= \phi\delta^n, \\ N_{n,b,a}\phi + N_{n,b,b}\psi &= -\alpha\phi\delta^n \end{aligned}$$

thus, by the analysis as in Theorem 2.6, we have

Theorem 2.12 *Suppose that* $(x,y,z) \in \tilde{S}_\alpha$. *We have*

(1) If $|\delta| < 1$, *then* $\Phi_\sigma^n(x,y,z) \to (2,2,2)$, $n \to \infty$. *In particular, if* $\delta = 0$, *then* $\Phi_\sigma^n(x,y,z) = (2,2,2)$;

(2) If $|\delta| = 1$, *then* $\Phi_\sigma^n(x,y,z) = (x,y,z)$.

By comparing with Theorem 2.6, we see that the conclusions of Theorem 2.12 are weaker than that of Theorem 2.6. In fact, it dose not tell us the behaviours of Φ_σ^n when $(x,y,z) \in \Omega_i \setminus \tilde{S}_\alpha$ and if $(x,y,z) \in \tilde{S}_\alpha$ but $|\delta| > 1$. To treat these cases, notice firstly that

$$\begin{pmatrix} N_{n,a,a}\phi + N_{n,a,b}\psi \\ N_{n,b,a}\phi + N_{n,b,b}\psi \end{pmatrix} = {}^tM_\sigma^n \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

where ${}^tM_\sigma$ is the transposition of M_σ. Secondly, if we note I^2 the unit square and we define $M_\sigma : I^2 \to I^2$ by $\overline{M_\sigma}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \equiv M_\sigma \begin{pmatrix} \phi \\ \psi. \end{pmatrix} \pmod 1$, then we have the following commutative diagram:

$$\begin{tikzcd} I^2 \arrow[r, "f"] \arrow[d, "\overline{M_\sigma}"'] & \Omega_i \arrow[d, "\Phi_\sigma"] \\ I^2 \arrow[r, "f"] & \Omega_i \end{tikzcd}$$

Figure 2

where f is the parameter map from I^2 to Ω_i.

On the other hand, we know that, if M_σ is invertible, then M_σ is ergodic, thus, if we note m the Lebesgue mesure, m the image of m by f, then $m-$a.e$(\phi,\psi) \in I^2$, the orbit of $M_\sigma^n \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is dense in I^2 and $m-$a.e$(x,y,z) \in \Omega_i$, $\Phi_\sigma^n(x,y,z)$ is dense in Ω_i $(n \geq 1)$. We obtain therefore

Theorem 2.13 *If* M_σ *is invertible, then for* $m-$*a.e*$(x,y,z) \in \Omega_i$, *the orbit of* $\Phi_\sigma(x,y,z)$ *is dense in* Ω_i, *that is* $\overline{orbit(\Phi_\sigma(x,y,z))} = \Omega_i$.

Remark 2.14 Theorem 2.13 gives an $m-$almost everywhere statement. The following example shows that the general case is very complex.

Let $\Delta > 1$ and $|\delta| < 1$ (i.e. Δ is Pisot number).

Let $V_1 = (V_1^1, V_1^2)$ and $V_2 = (V_2^1, V_2^2)$ be the eigenvectors corresponding to the eigenvalues Δ and δ respectively. If $V \in \mathbb{R}^2$ is a vector, then there are two numbers $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $V = \alpha_1 V_1 + \alpha_2 V_2$. By the Perron-Frobenius Theorem, $V_1 = (V_1^1, V_1^2)$ is positive, since $M_\sigma^n V = \alpha_1 \Delta^n V_1 + \alpha_2 \delta^n V_2$, notice that $\alpha_2 \delta^n V_2 \to 0$, we have therefore $M_\sigma^n V \sim \alpha_1 \Delta^n V_1$, $n \to \infty$.

Let $\Gamma = \{(u,v) \in I^2;\ v/u = V_1^2/V_1^1\}$, then Γ is a family of parallel segments with slope V_1^2/V_1^1 situated in I^2 which may be obtained by by the straight line $v = (V_1^2/V_1^1)u$ by taking module 1.

If $V_1^2/V_1^1\} \in \mathbb{Q}$, then Γ is an union of a finite number segment, otherwise it is an union of infinity segments which is dense in I^2.

Now let $V_1^2/V_1^1\} \in \mathbb{Q}$, and let $\alpha_1 \in \mathbb{Q}$, $\Delta \in \mathbb{N}$, then $\{\alpha_1 \Delta^n V_1\}_{n \geq 1}$ is a finite set of Γ which attracts all the points with $\alpha_1 \in \mathbb{Q}$.

If $\Delta \in \mathbb{N}$, then, we know that by a classic result, for almost all $\alpha_1 \in \mathbb{R}$, the sequence $\overline{\{\alpha_1 \Delta^n V_1\}_{n\geq 1}} = \Gamma$ and Γ is an attractor.

The general case is rather difficult, in fact, it involves the arithmetic properties of the sequence $\{\alpha_1 \Delta^n\}_{n \geq 1}$. For example, we do not know if the sequence $\{(3/2)^n\}_{n\geq 1}$ is equidistributed mod 1, we do not know even whether this sequence is dense in I.

3 Example

In this section, we observe three typical substitution.

Example 1 $\sigma(a) = ab$, $\sigma(b) = a$ (Fibonacci substitution)

In this case, we have

$$\overline{M_\sigma} = M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Delta = \frac{(\sqrt{5}+1)}{2}, \quad \delta = \frac{(\sqrt{5}-1)}{2}, \quad \alpha = \frac{d_a}{d_b} = \frac{(\sqrt{5}+1)}{2}.$$

By Table 1, $\Phi_\sigma(p_{00}) = p_{00}$, $\Phi_\sigma(p_{01}) = p_{10}$, $\Phi_\sigma(p_{10}) = p_{11}$, $\Phi_\sigma(p_{11}) = p_{01}$, that is, p_{00} is a fixed point of Φ_σ, and p_{01}, p_{10}, p_{11} are period-3 points.

By Proposition 2.1, we have

$$\Phi_\sigma(\Omega_0^{00}) \subset \Omega_0^{00}, \quad \Phi_\sigma(\Omega_0^{01}) \subset \Omega_0^{10}, \quad \Phi_\sigma(\Omega_0^{10}) \subset \Omega_0^{11}, \quad \Phi_\sigma(\Omega_0^{11}) \subset \Omega_0^{01}$$

By Theorem 2.6, all points of S_α are attracted by Ω_S and all point of $\Omega_0^{00} \setminus S_\alpha$ tend to infinity.

If $(x,y,z) \in \tilde{S}_\alpha$, then $\Phi_\sigma^n(x,y,z) \to (2,2,2)$ by Theorem 2.12. In particular, $\{\Phi_\sigma^n\}_{n\geq 1}$ is dense in Ω_i.

Example 2 $\sigma(a) = ab$, $\sigma(b) = ba$ (Thue-Morse substitution)

We have

$$\overline{M_\sigma} = M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Delta = 2, \quad \delta = 0, \quad \alpha = 1.$$

By Table 1, $\Phi_\sigma(p_{00}) = p_{00}$, $\Phi_\sigma(p_{01}) = p_{11}$, $\Phi_\sigma(p_{10}) = p_{11}$, $\Phi_\sigma(p_{11}) = p_{00}$. Thus for $n \geq 3$, $\Phi_\sigma^n(p_{\epsilon\eta}) = p_{00}$, and $\Phi_\sigma^n(\Omega_0^{\epsilon\eta}) \subset \Omega_0^{00}$.

If $(x,y,z) \in S_\alpha$, then $\Phi_\sigma(x,y,z) = (x,y,z)$, otherwise $\Phi_\sigma^n(x,y,z) \to (\infty,\infty,\infty)$.

Since $N_{n,a,a} = N_{n,a,b} = N_{n,b,a} = N_{n,b,b} = 2^n$, hence for $(x,y,z) \in \Omega_i$,

$$\begin{aligned} \Phi_{\sigma,a}^n(x,y,z) &= 2\cos(2^n(\phi+\psi)) \\ \Phi_{\sigma,b}^n(x,y,z) &= 2\cos(2^n(\phi+\psi)) \end{aligned}$$

thus, if $\phi + \psi = 0$, i.e. $(x,y,z) \in \tilde{S}_\alpha$, then $\Phi_\sigma(x,y,z) = (2,2,2)$; if $\phi+\psi \in \mathbb{Q}$, then Φ_σ^n admits a periodic orbit and if $\phi + \psi \in \mathbb{R} \setminus \mathbb{Q}$, then $\overline{\{\Phi_\sigma^n\}_{n\geq 1}} = \{(x,y,z) \in \Omega_i;\ x = y\}$.

Example 3 $\sigma(a) = ab$, $\sigma(b) = aa$ (Toeplitz substitution)

We have

$$M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \overline{M_\sigma} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Delta = 2, \quad \delta = -1, \quad \alpha = 2.$$

From Table 1, we get $\Phi_\sigma(p_{00}) = p_{00}$, $\Phi_\sigma(p_{01}) = p_{10}$, $\Phi_\sigma(p_{10}) = p_{10}$, $\Phi_\sigma(p_{11}) = p_{00}$. Thus p_{00} and p_{10} are fixed points. Furthermore,

$$\Phi_\sigma(\Omega_0^{00}) \subset \Omega_0^{00}, \quad \Phi_\sigma(\Omega_0^{11}) \subset \Omega_0^{00}, \quad \Phi_\sigma(\Omega_0^{01}) \subset \Omega_0^{10}, \quad \Phi_\sigma(\Omega_0^{10}) \subset \Omega_0^{10}$$

If $(x,y,z) \in S_\alpha$, then $\Phi_\sigma(x,y,z) = (x,y,z)$, that is, all the points of S_α are fixed points. By Theorem 2.12, we have also for $(x,y,z) \in \tilde{S}_\alpha$, $\Phi_\sigma(x,y,z) = (x,y,z)$.

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