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The factor composition matrix of sequences☆

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Abstract

Let $S=\{a,b\}$ be a two-letter alphabet and s a sequence over S. An infinite matrix $(t_{i,j})_{i,j\geqslant 0}$ is associated with s in the following way: $t_{i,j}=1$ if s has a factor which contains i times a and j times b; otherwise $t_{i,j}=0$. This matrix will be called the factor composition matrix (FCM) of the sequence s. In this paper, combinatorial properties of certain sequences are studied via their FCMs. In particular

(i) the FCM of the Thue–Morse sequence is shown to be pentadiagonal, and substitutions whose fixed points have the same FCM as the Thue–Morse sequence are determined;
(ii) an algorithm for computing the FCM of a sturmian sequence is presented;
(iii) the FCMs of fixed points of invertible substitutions are characterized in terms of their singular decompositions.

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0. Introduction

Combinatorial properties of words and sequences are of increasing importance in various fields of mathematics and computer science as well as of physics, biology ... . They have been extensively studied by using various methods and techniques (for instance, see [1,3,7,8,10], and the references therein). Here we present a new aspect of the combinatorics of words.

Let $S = \{a, b\}$ be a two-letter alphabet. Let S^* and $\tilde{S}$ stand respectively for the free monoid and the free group generated by S. The empty word ε is their identity element. Let S^ω be the set of sequences (or infinite words), indexed by $\mathbb{N}$, on S.

If $w \in S^*$ is a word, we denote by $|w|$ its length and by $|w|_a$ (resp. $|w|_b$) the number of times the letter a (resp. b) appears in it. Let $L(w)$ stand for the vector $(|w|_a, |w|_b)$.

A word v is a factor of a word w, and then we write $v \prec w$, if there exists $u, u' \in S^*$, such that $w = uvu'$. We say that v is a prefix (resp. suffix) of a word w, and then we write $v \lhd w$ (resp. $v \rhd w$), if there exists $u \in S^*$ such that $w = vu$ (resp. $w = uv$). The notions of prefix and factor extend in a natural way to infinite words.

Let $s \in S^\omega$ be a sequence, we define an infinite matrix $(t_{i,j})_{i,j \geqslant 0}$ as follows:

$$t_{i,j} = \begin{cases} 1 & \text{if there exists } w \prec s \text{ such that } L(w) = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $t_{0,0} = 1$. We call this matrix the factor composition matrix (FCM) of the sequence s.

If σ is a primitive substitution on a two-letter alphabet, its FCM is the FCM of any fixed point of σ or σ^2.

This paper is divided into five sections besides this introduction. Section 1 contains the definitions, notations, and basic results needed in the latter sections. In the second one, it is shown that the 1's in the FCM of Thue–Morse sequence exactly form five diagonals. In this section we determine the substitutions which have the same FCM as the Thue–Morse sequence. Section 3 is devoted to FCMs of sturmian sequences. The case of the Fibonacci sequence is studied in Section 4. In the last section, we generalize the results of the fourth one to fixed points of invertible substitutions by using their singular decompositions. It is shown that the 1's in the corresponding FCM are generated by a substitution acting on certain patterns. Examples are given at the end.

1. Definitions and notations

We shall use the following terminology and notations.

- Let $w = x_1 x_2 \cdots x_n \in S^*$. The reversed word $\overleftarrow{w}$ of w is defined to be $\overleftarrow{w} = x_n \cdots x_2 x_1$. A word w is called a palindrome if $w = \overleftarrow{w}$. The set of palindromes is denoted by $\mathfrak{P}$.
- A morphism $\tau : S^* \to S^*$ is called a substitution of S^*. In this work, we deal only with *non-erasing* substitutions, which means that $\tau(a)$ and $\tau(b)$ are different from ε. We denote by F_τ any one of the fixed points of τ (i.e., F_τ is an infinite sequence such that $\tau(F_\tau) = F_\tau$), if it exists, and by M_τ the matrix $(L(\tau(a))^t, L(\tau(b))^t)$ called the matrix of

the substitution τ (it is also called incidence or transition matrix). A substitution is said to be primitive if its matrix is.

- Let $w = x_1x_2 \cdots x_n \in \tilde{S}$, we denote by w^{-1} the inverse of w, that is $w^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}$. When $w = uv$, with $u \in S^*$ and $v \in S^*$, it will be convenient to write $u = wv^{-1}$ and $v = u^{-1}w$.
- Let $w \in S^*$, we denote by $\imath_w$ the inner isomorphism $u \mapsto wuw^{-1}$, $u \in S^*$. If there exists a $w \in S^*$ such that $\phi = \imath_w \tau$ or $\tau = \imath_w \phi$, we say that ϕ is conjugate to τ and write $\phi \sim \tau$.
- Let $s_1, s_2 \in S^\omega$ be two sequences. We say that s_1 and s_2 are locally isomorphic, and then we write $s_1 \simeq s_2$, if, for any factor w of s_1, w or $\overleftarrow{w}$ is a factor of s_2 and vice versa. Obviously, two locally isomorphic sequences have the same FCM.

If a primitive substitution has two fixed points, these fixed points are easily seen to be locally isomorphic: indeed, they have the same factors.

We have the following lemma [11].

Lemma 1.1. *Let τ_1, τ_2 be two substitutions with fixed points F_{τ_1}, F_{τ_2}, respectively. If $\tau_1 \sim \tau_2$, then $F_{\tau_1} \simeq F_{\tau_2}$.*

- A sequence $s \in S^\omega$ is called sturmian if, for any $n \in \mathbb{N}$, the number of its factors of length n is exactly $n + 1$. The sequence s is said to be balanced if, for any pair $u,\ v$ of factors of s with the same length, one has $\big||u|_a - |v|_a\big| \leqslant 1$. The sequence s is ultimately periodic, if there exist $u, v \in S^*$ such that $s = uvvvv \cdots = uv^\infty$. We have the following proposition (see for example [1,6]).

Proposition 1.2. *A sequence is sturmian if and only if it is balanced and not ultimately periodic.*

From the definition, we know that for a sturmian sequence, for any $n \in \mathbb{N}$, there exists a unique factor w of length n such that both aw and bw are factors of this sequence. These words w are called (left) special words.

A special (or characteristic) sturmian sequence is a sturmian sequence whose all prefixes are special words. Other equivalent definitions and properties of sturmian sequences can be found in [1,2,4].

- We denote by Aut($\tilde{S}$) the group of automorphisms of $\tilde{S}$. It is known [8,9] that Aut($\tilde{S}$) is generated by the following three special automorphisms

$$\sigma = (ab, a), \quad \pi = (b, a), \quad \psi = (a, b^{-1}), \tag{1}$$

  where $\varrho = (u, v)$ means that $\varrho(a) = u$ and $\varrho(b) = v$.
- If a substitution τ is also in Aut($\tilde{S}$), it is called an *invertible substitution*. The set of invertible substitutions is denoted by IS(S^*). This set can be characterized by the following result [12].

Proposition 1.3. IS(S^*) *is generated by the three substitutions $\pi = (b, a)$, $\sigma = (ab, a)$, and $\rho = (ba, a)$*: IS$(S^*) = \langle (b, a), (ab, a), (ba, a) \rangle$.

In particular, this implies that a morphism of the free monoid on two letters is sturmian if and only if it is invertible [7,12].

The following facts about invertible substitutions will play an important role in the latter sections [11–13].

For each invertible substitution τ, there exists $\varphi \in \langle \pi, \sigma \rangle$ (where π and σ are defined by (1)) such that $M_\varphi = M_\tau$. Two invertible substitutions τ_1 and τ_2 are conjugate if and only if $M_{\tau_1} = M_{\tau_2}$. So, due to Lemma 1.1, to study the FCMs of invertible substitutive sequences, that is the fixed points of invertible substitutions, it is enough to consider the substitutions in $\langle \pi, \sigma \rangle$.

Define $\mathcal{G}_\sigma$ to be the set of the following substitutions:

$$\tau = \sigma^{n_p} \circ \pi \circ \sigma^{n_{p-1}} \circ \pi \cdots \sigma^{n_2} \circ \pi \circ \sigma^{n_1}, \tag{2}$$

where $n_j \geqslant 1$ for $1 \leqslant j \leqslant p$, then we have [11]

Proposition 1.4. *If $\tau \in \mathcal{G}_\sigma$, any fixed point of τ is special sturmian.*

2. The Thue–Morse sequence and the "five-diagonal property"

The substitution $\tau = (ab, ba)$ is called the Thue–Morse substitution. It has two fixed points and any one of them is called Thue–Morse sequence. Denote by F_τ the Thue–Morse sequence with prefix a. We will show that the 1's in its FCM form exactly five diagonals.

Proposition 2.1. *For the Thue–Morse sequence, we have*

$$t_{i,j} = 1 \quad \textit{if and only if} \quad |i - j| \leqslant 2. \tag{3}$$

In the proof of this proposition, we need the following lemma.

Lemma 2.2. *Let $s = s_0 s_1 s_2 s_3 \cdots \in S^\omega$ be a sequence which is not ultimately periodic. Then for every natural number n, there exists a factor w (resp. v) of length n such that $awb \prec s$ (resp. $bva \prec s$).*

Proof. Suppose the conclusion does not hold, then there is a $n \in \mathbb{N}$ such that

$$(s_k = a) \Rightarrow (s_{k+(n+1)} = a).$$

Let us recursively define two sequences E_j and m_j in the following way: $m_0 = 0$, $E_0 = \mathbb{N}$; if the set $\{i \in E_j \mid s_i = a\}$ is empty, then we stop, if not m_{j+1} is its smallest element and $E_{j+1} = E_j \setminus \big(m_{j+1} + (n+1)\mathbb{N}\big)$. This process lasts at most $n+1$ steps. It is clear that following the last n_j, the sequence is periodic of period $(n+1)$. $\square$

Proof of Proposition 2.1. It is easy to see that $t_{i,j} = 1$ implies $|i - j| \leqslant 2$. We wish to show that $|i - j| \leqslant 2$ implies $t_{i,j} = 1$.

Since the Thue–Morse sequence F_τ is not ultimately periodic, for any natural number i, there exists a word w of length i such that $awb \prec F_\tau$. Then $\tau(awb) = ab\tau(w)ba$ is a factor of F_τ as well as $b\tau(w)b$ and $b\tau(w)$. As $\tau(w)$ contains as many a's as b's (indeed, i of each), this shows that $t_{i,i+1} = t_{i,i+2} = 1$. For a similar reason $t_{i+1,i} = t_{i+2,i} = 1$. $\square$

Definition 2.3. We say that a sequence has the five-diagonal property provided it has the same FCM as the Thue–Morse sequence. We also say a non-identity substitution has the five-diagonal property if all its fixed points have.

We need some lemmas.

Lemma 2.4. *Let $s \in S^\omega$ be a sequence. If s has the five-diagonal property, then s is not periodic.*

Proof. Suppose that s is periodic. Then $s = u^\infty$ with $u \in S^*\backslash\{\varepsilon\}$. For every factor w of s of length $|u|$ we have $(|w|_a, |w_b|) = (|u|_a, |u_b|)$, thus the set $\{(i, j) : t_{i,j} = 1, i + j = |u|\}$ is a singleton, which is a contradiction. $\square$

Lemma 2.5. *Let τ be a substitution. If τ has the five-diagonal property, then*

$$|\tau(a)|_a = |\tau(a)|_b \quad \textit{and} \quad |\tau(b)|_a = |\tau(b)|_b.$$

Proof. The proof will be completed in three steps.

Step 1. We prove first that $|\tau(ab)|_a = |\tau(ab)|_b$.

Consider a factor w of length 16 of F_τ. Set $|w|_a = 8 + k$ and $|w|_b = 8 - k$. It results from the five-diagonal property that $|k| \leqslant 1$ (indeed, $\big| |w|_a - |w|_b \big| \leqslant 2$).

We have $L(\tau(w)) = 8\,L(\tau(ab)) + kL(\tau(a)) - kL(\tau(b))$, so

$$2 \geqslant \big| |\tau(w)|_a - |\tau(w)|_b \big| \geqslant 8 \left(\big| |\tau(ab)|_a - |\tau(ab)|_b \big|\right) - 2 - 2.$$

This last inequality implies $|\tau(ab)|_a = |\tau(ab)|_b$, as stated.

Step 2. If $k = |\tau(a)|_a - |\tau(a)|_b$, we shall show that $|k| \leqslant 1$.

Due to step 1, we have $|\tau(b)|_a - |\tau(b)|_b = -k$, which implies $\big| |\tau^2(a)|_a - |\tau^2(a)|_b \big| = k^2$. But $\tau^2(a)$ is a factor of F_τ, so we have $k^2 \leqslant 2$.

Step 3. Here, we prove that $k = 0$.

Assume that $k = 1$, we have

1. $aa \not\prec \tau(a)$

   Otherwise $\tau(a) = uaav$, where $u, v \in S^*$. Since $k = 1$, we have $|u|_a + |v|_a + 2 - |u|_b - |v|_b = 1$. Since $aa \prec F_\tau$, $\tau(aa) = uaavuaav$ is a factor of F_τ. For the factor $w = aavuaa \prec F_\tau$, we have $|w|_a - |w|_b = 3$, which contradicts the five-diagonal property.
2. $bb \not\prec \tau(a)$

Otherwise $\tau(a) = ubbv$. From 1, $aa \not\lhd \tau(a)$, then $aa \not\lhd u$ and $|u|_a - |u|_b \leqslant 1$. For the same reason $|v|_a - |v|_b \leqslant 1$, then $|\tau(a)|_a - |\tau(a)|_b \leqslant 1 + 1 - 2 = 0$, and a contradiction occurs again.

From 1 and 2, we have $\tau(a) = abab \cdots a = (ab)^n a$, and for the same reason, $\tau(b) = baba \cdots b = (ba)^m b$. Thus $F_\tau = (ab)^\infty$ is periodic, and the 1's in $(t_{i,j})$ form only three diagonals.

In the same way, one can show that $k = -1$ is not possible either. □

Remark 2.6. It results from Lemma 2.5, that if a substitution has the five-diagonal property, it must be primitive and non-invertible.

Remark 2.7. A primitive substitutive sequence is ultimately periodic if and only if it is periodic because of the minimality (see Lemma 7 in [5]). Therefore the fixed point of a substitution having the five-diagonal property is not ultimately periodic.

The next lemma is readily checked.

Lemma 2.8. *Let τ be a substitution.*

(i) *If τ has the five-diagonal property, then every factor w of $\tau(aa)$, $\tau(ab)$, $\tau(ba)$ or $\tau(bb)$ satisfies $||w|_a - |w|_b| \leqslant 2$.*

(ii) *If any factor w of $\tau(aa)$, $\tau(ab)$, $\tau(ba)$, and $\tau(bb)$ satisfies $||w|_a - |w|_b| \leqslant 2$, $|\tau(a)|_a = |\tau(a)|_b$, and $|\tau(b)|_a = |\tau(b)|_b$, then the 1's in the FCM of τ must be all on the five diagonals (i.e., $t_{i,j} = 1 \Rightarrow |i - j| \leqslant 2$).*

Without loss of generality, we always assume that $a \lhd \tau(a)$, thus τ has a fixed point $F_\tau = \lim_n \tau^n(a)$.

The following theorem describes the substitutions τ satisfying the five-diagonal property.

Theorem 2.9. *Let τ be a substitution such that $a \lhd \tau(a)$, then τ has the five-diagonal property if and only if F_τ is not ultimately periodic and τ is of one of the following types*:

Type 1 $\begin{cases} \tau(a) = U_0 U_1 \cdots U_n, \\ \tau(b) = V_0 V_1 \cdots V_m, \end{cases}$

Type 2 $\begin{cases} \tau(a) = (ab)^k, \\ \tau(b) = (ab)^l aab V_0 V_1 \cdots V_m b, \end{cases}$

Type 3 $\begin{cases} \tau(a) = (ab)^l aab U_0 U_1 \cdots U_n b, \\ \tau(b) = a V_0 V_1 \cdots V_m b, \end{cases}$

where $U_i, V_j \in \{ab, ba\}$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$, $k \geqslant 1$, $l \geqslant 0$, and $U_0 = ab$.

Proof. *Necessity*: Suppose that τ has the five-diagonal property. Then its fixed point F_τ is not ultimately periodic.

By Lemma 2.5, $|\tau(a)|$ and $|\tau(b)|$ are even and we can write

$$\tau(a) = U_1 U_2 \cdots U_n, \tau(b) = V_1 V_2 \cdots V_m,$$

where $U_i, V_j \in \{aa, ab, ba, bb\}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $a \lhd U_1$.

There are three possibilities which match the types in the statement of the theorem.

Type 1—$U_i, V_j \in \{ab, ba\}$ for all i, j.

Type 2—$U_i \in \{ab, ba\}$ for all i, and at least one V_j equals aa.

Since $|\tau(b)|_a = |\tau(b)|_b$, there exists V_k which is equal to bb. Set $r = \min\{j : V_j = aa\}$, $s = \min\{j : V_j = bb\}$.

We claim that $r < s$. Indeed if we had $s < r$, we would have $\tau(b) = V_1 V_2 \cdots V_{s-1} bb V_{s+1} \cdots V_m$, with $V_j \in \{ab, ba\}$ for $1 \leqslant j \leqslant s-1$. It would follow

$$\tau(ab) = U_1 U_2 \cdots U_n V_1 V_2 \cdots V_{s-1} bb V_{s+1} \cdots V_m$$

with $a \lhd U_1$. Then we would have $w = a^{-1} U_1 U_2 \cdots U_n V_1 V_2 \cdots V_{s-1} bb \prec \tau(ab)$ and $|w|_b - |w|_a = 3$, which would contradict Lemma 2.8(i). Therefore we have $\tau(b) = V_1 V_2 \cdots V_{r-1} aa V_{r+1} \cdots V_m$ with $V_j \in \{ab, ba\}$ for $1 \leqslant j \leqslant r-1$.

Now we can show that $U_i = ab$ for all i and $V_j = ab$ for $1 \leqslant j \leqslant r-1$. In fact, if we had $U_i = ba$, then we would have $w = b^{-1} U_i U_{i+1} \cdots U_n V_1 V_2 \cdots V_{r-1} aa \prec \tau(ab)$, and $|w|_a - |w|_b = 3$, which is a contradiction. For the same reason, $V_j = ab \quad (1 \leqslant j \leqslant r-1)$.

Now, we know that $\tau(b) = (ab)^{r-1} aa V_{r+1} \cdots V_m$. Obviously, $b \lhd V_{r+1}$, so we can write $V_{r+1} \cdots V_m = b W_1 W_2 \cdots W_{m-r-1} \alpha$, where $W_i \in \{aa, ab, ba, bb\}$ and $\alpha \in \{a, b\}$. We wish to show that $W_i \in \{ab, ba\}$ for all i and $\alpha = b$. Suppose that one of the W_i equals aa or bb and set $k = \min\{i : w_i = aa \quad \text{or} \quad w_i = bb\}$. If $W_k = aa$, then $w = aab W_1 W_2 \cdots W_{k-1} aa \prec \tau(b)$ and $|w|_a - |w|_b = 3$; this is a contradiction. If $W_k = bb$ then $w = b W_1 W_2 \cdots W_{k-1} bb \prec \tau(b)$ and $|w|_b - |w|_a = 3$, which again is a contradiction.

Thus $W_i \in \{ab, ba\}$ and α must be b.

Type 3—For some k, $U_k = aa$.

We argue as for type 2: we can write

$$\tau(a) = (ab)^{k-1} aab W_1 W_2 \cdots W_{n-k-1} b, \tau(b) = a X_1 X_2 \cdots X_{m-1} b,$$

where $W_i, X_j \in \{ab, ba\}$ for all i, j.

Sufficiency: Proof for Type 1. It results from Lemma 2.8(ii) that all 1's of $(t_{i,j})$ are in the five diagonals. We only need to show that $|i - j| \leqslant 2$ implies $t_{i,j} = 1$.

Consider a two-letter alphabet $\{A, B\}$ and the morphism $\phi : \{A, B\}^* \to \{a, b\}^*$ defined by $A \mapsto ab$, $B \mapsto ba$. This morphism has a natural extension to $\{A, B\}^\omega$ which still be denoted by ϕ.

Consider the substitution

$$\gamma = (X_0 X_1 \cdots X_n Y_0 Y_1 \cdots Y_m, Y_0 Y_1 \cdots Y_m X_0 X_1 \cdots X_n)$$

on $\{A, B\}^*$, where $X_i = \phi^{-1}(U_i)$, $Y_i = \phi^{-1}(V_i)$.

We have that $\tau\phi = \phi\gamma$, from which the equality $\phi\gamma^n = \tau^n\phi$ results by induction. Hence (notice that $U_0 = ab$)

$$\begin{aligned}\phi(F_\gamma) &= \phi(\lim_{n\to\infty}\gamma^n(A)) = \lim_{n\to\infty}\phi\gamma^n(A)\\ &= \lim_{n\to\infty}\tau^n\phi(A) \quad = \lim_{n\to\infty}\tau^n(ab) = F_\tau.\end{aligned}$$

Since F_τ is not ultimately periodic, F_γ is not either. Then it results from Lemma 2.2 that, for any natural number i, there exits a word $W \in \{A, B\}^*$ of length i such that $AWB \prec F_\gamma$. Then $b\phi(W)b \prec F_\tau$, which means $t_{i,i+2} = 1$. We have $t_{i+2,i} = 1$ for the same reason. The assertion $|i - j| \leqslant 1 \Rightarrow t_{i,j} = 1$ is trivial.

Proof for Type 2. Substitution τ is easily seen to be conjugate to

$$\big((ba)^k, (ba)^{l+1}abV_0V_1\cdots V_m\big),$$

which is of type 1. We conclude by using Lemma 1.1.

Proof for Type 3.

This time τ is conjugate to $\big((ba)^{l+1}abU_0U_1\cdots U_n, baV_0V_1\cdots V_m\big)$. □

Remark 2.10. This theorem answers partially the following more general problem: to characterize the sequences having the five-diagonal property.

3. The FCMs of sturmian sequences

Proposition 3.1. *Let s be a sturmian sequence. Then for any $n \geqslant 1$, there are two and only two* 1's *among* $\{t_{i,j}\}_{0\leqslant i,j\leqslant n,\ i+j=n}$ *and the two* 1's *are adjacent.*

Proof. If for one $n \geqslant 1$, there were only one i such that $0 \leqslant i \leqslant n$ and $t_{i,n-i} = 1$, then the sequence would be periodic of period n. Therefore, the assumption that s is not ultimately periodic implies that there are at least two 1's in $\{t_{i,j}\}_{0\leqslant i,j\leqslant n,\ i+j=n}$. Since s is balanced there are only two 1's and they are adjacent. □

Remark 3.2. The converse is not true: the condition "there are two and only two 1's among $\{t_{i,j}\}_{0\leqslant i,j\leqslant n,\ i+j=n}$ and the two 1's are adjacent" can merely guarantee the fact that the original sequence is balanced (for instance, consider the sequence ba^∞).

Proposition 3.3. *For the FCM,* $\{t_{i,j}\}$, *of a sturmian sequence s, we have*

(i) $\sum_j t_{i,j} \geqslant 2$ *for* $i \geqslant 0$, *and* $\sum_i t_{i,j} \geqslant 2$ *for* $j \geqslant 0$;
(ii) *If* $t_{i,j} = t_{i-1,j+1} = 1$, *then* $t_{i,j+1} = 1$;
(iii) *If there exists* $k \geqslant 2$ *such that* $t_{i,j} = 1$ *and* $t_{i+k,j} = 1$ (*resp.* $t_{i,j+k} = 1$), *then, for any* $1 \leqslant l \leqslant k-1$, *one has* $t_{i+l,j} = 1$ (*resp.* $t_{i,j+l} = 1$);
(iv) *If* $aa \not\prec s$ *then* $\sum_i t_{i,j} \leqslant 4$, *and, if* $bb \not\prec s$, *then* $\sum_j t_{i,j} \leqslant 4$.

Proof. (i) Suppose that $\sum_j t_{i_0,j} = 1$. This can happen only with $i_0 \geqslant 1$. Let j_0 and w be such that $t_{i_0,j_0} = 1$, $w \prec s$, and $L(w) = (i_0, j_0)$.

We have $wa \prec s$ ($wb \prec s$ is impossible, since $t_{i_0,j_0+1} \neq 1$). As $j_0 > 0$ or $t_{i_0,j_0-1} \neq 1$, we have $b \ntriangleleft w$, so $a \triangleleft w$. Furthermore, $a^{-1}wa \prec s$ with $|a^{-1}wa|_a = i_0$. Repeating this argument shows that $w = a^{i_0}$. Due to Proposition 3.1, the only factors of length i_0 which can appear are a^{i_0} and $a^j b a^{i_0-j-1}$ with $0 \leqslant j < i_0$. This implies that s is periodic after the first appearance of the factor a^{i_0}. This is a contradiction.

(ii) Suppose that $t_{i,j+1} = 0$.

If $w \prec s$ with $L(w) = (i, j)$, then $wb \not\prec s$ and $wa \prec s$. If $b \triangleleft w$, then $L(b^{-1}wa) = (i+1, j-1)$ and $t_{i+1,j-1} = 1$. But $t_{i-1,j+1} = 1$, this contradicts the fact that s is balanced. So we have $a \triangleleft w$. The proof ends as in (i).

(iii) By induction on k. The result holds for $k = 1$.

Suppose that the assertion holds for $k < n$ ($n \geqslant 2$) and does not hold for $k = n$. Thus there exists (i, j) such that $t_{i,j} = t_{i+n,j} = 1$ and $t_{i+l,j} = 0$ for $1 \leqslant l < n$.

Take $w \prec s$ such that $L(w) = (i+n, j)$. Since $t_{i+(n-1),j} = 0$, a can be neither a prefix nor a suffix of w, so there exists v, such that $w = bvb$. For any factor v' of v, we have $|v'|_b \leqslant |v|_b = j - 2$. But $t_{i,j} = 1$, this again contradicts the fact that s is balanced.

(iv) Suppose that $\sum_i t_{i,j} \geqslant 5$. Then, due to (iii), there exists i such that $t_{i+k,j} = 1$ for $0 \leqslant k \leqslant 4$. Consider $w \prec s$ such that $L(w) = (i+4, j)$ and write $w = vv'$, with $|v| = 4$. We have $|v'| = i + j$ and $|v'|_b = j - |v|_b$. As $t_{i,j} = 1$ and s is balanced, we have $|v|_b \leqslant 1$. This proves that aa is a factor of v, and therefore of s. □

Now, if $s = s_0 s_1 \cdots s_n \cdots$, we set $s(n) = s_0 s_1 \ldots s_{n-1}$ and define two sequences

$$a_k = L(s(k)) + (1, 0) \quad \text{and} \quad b_k = L(s(k)) + (0, 1).$$

Theorem 3.4. *Let s be special sturmian. Then $t_{i,j} = 1$ if and only if $(i, j) \in \{a_k : k \geqslant 0\} \bigcup \{b_k : k \geqslant 0\}$.*

Proof. Since s is special sturmian, one has $as(n) \prec s$ and $bs(n) \prec s$. Thus $(i, j) = a_k$ or b_k implies $t_{i,j} = 1$. We obtain “if”. From Proposition 3.1, we obtain “only if”. □

Now, for a sturmian sequence s, we define a new sequence $\tilde{s} = \tilde{s}_0 \tilde{s}_1 \cdots \tilde{s}_n \cdots$ as follows.

By Proposition 3.1, for $n \geqslant 1$, there are only two 1’s, which are adjacent, in $\{t_{i,j}\}_{i+j=n}$; denote them by $t_{i,j}$ and $t_{i-1,j+1}$. It results from Proposition 3.3(2) that $t_{i,j+1} = 1$. Thus there is only one 1 among $t_{i+1,j}$ and $t_{i-1,j+2}$. We define

$$\tilde{s}_{n-1} = \begin{cases} a & \text{if } t_{i+1,j} = 1, \\ b & \text{if } t_{i-1,j+2} = 1. \end{cases}$$

We have the following proposition.

Proposition 3.5. *If s be a special sturmian sequence, then $\tilde{s} = s$.*

Proof. This results from the definitions of a_k, b_k, $\tilde{s}$ and from Theorem 3.4. $\square$

Theorem 3.6. *For a sturmian sequence s, $\tilde{s}$ is the special sequence of s.*

Proof. Let g be the special sequence of s, then $g \simeq s$, and they have the same $(t_{i,j})$. The conclusion follows from Proposition 3.5. $\square$

Now, we are going to rephrase in a more visual way the properties of the FCMs of special sturmian sequences. Set

$$\bar{a} = \begin{array}{cc} & \mathbf{1} \\ 1 & * \end{array}, \quad \bar{b} = \begin{array}{cc} & 1 \\ \mathbf{1} & * \end{array}, \quad \text{and} \quad \bar{c} = \begin{array}{cc} \mathbf{1} & 1 \\ 1 & * \end{array}$$

and define $\bar{a}\bar{a}$, $\bar{a}\bar{b}$, $\bar{b}\bar{a}$, $\bar{b}\bar{b}$, $\bar{c}\bar{a}$, and $\bar{c}\bar{b}$ to be respectively

$$\begin{array}{cc} & \mathbf{1} \\ 1 & 1 \\ 1 & * \end{array}, \quad \begin{array}{ccc} & \mathbf{1} & 1 \\ 1 & 1 & * \end{array}, \quad \begin{array}{cc} & 1 \\ \mathbf{1} & 1 \\ 1 & * \end{array}, \quad \begin{array}{ccc} & 1 & 1 \\ \mathbf{1} & 1 & * \end{array}. \quad \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & * \end{array}, \quad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & * \end{array}$$

(the boldface 1 of the second factor replaces the asterisk of the first one). In the same way

$$\bar{a}\bar{c} = \begin{array}{ccc} & \mathbf{1} & \\ 1 & 1 & 1 \\ & 1 & * \end{array}, \quad \bar{b}\bar{c} = \begin{array}{ccc} & 1 & \\ \mathbf{1} & 1 & 1 \\ & 1 & * \end{array}, \quad \text{and} \quad \bar{c}\bar{c} = \begin{array}{ccc} \mathbf{1} & 1 & \\ 1 & 1 & 1 \\ & 1 & * \end{array}.$$

We can iterate this multiplication. For instance,

$$\bar{a}\bar{b}\bar{a} = \begin{array}{ccc} & \mathbf{1} & 1 \\ 1 & 1 & 1 \\ & 1 & * \end{array}, \quad \bar{a}\bar{b}\bar{a}\bar{a} = \begin{array}{ccc} & \mathbf{1} & 1 \\ 1 & 1 & 1 \\ & 1 & 1 \\ & 1 & * \end{array}, \quad \text{and} \quad \bar{a}\bar{b}\bar{a}\bar{a}\bar{b} = \begin{array}{cccc} & \mathbf{1} & 1 & \\ 1 & 1 & 1 & \\ & 1 & 1 & 1 \\ & 1 & 1 & * \end{array}.$$

A more formal definition of this monoid of patterns is given a few lines below.

It is convenient to write $\overline{abaab} = \bar{a}\bar{b}\bar{a}\bar{a}\bar{b}$, for instance.

Now, if s is a special sturmian sequence, the pattern $\bar{c}\,\overline{s(n)}$ is the beginning (i.e., the pattern at the left upper corner) of the FCM of s. This is a mere reformulation of the results of this section.

3.1. *Formal definition of the monoid of patterns*

Let M be a monoid, G a group, and X a space on which G acts to the right (all the actions and composition laws are noted multiplicatively). We endow the set $X^M \times G$ with the following multiplication: $(u, \alpha)(v, \beta) = (w, \gamma)$, where w is defined by $w(x) = u(x)v(x\alpha)$. It is easily checked that this operation is associative. In other terms, as an element α of G can be considered as an automorphism of the monoid X^M ($\alpha(u)(x) = u(x\alpha)$), the newly defined monoid is exactly the semi-direct product of X^M and G.

In the present case, we take $X = G = \mathbb{Z}^2$ and $M = \{0, 1\}$ endowed with the "or" operation $\vee$, the action of $\alpha \in G$ on $x \in X$ being $x - \alpha$. We are going to identify a pattern, as defined above, with an element (u, α) of $X^M \times G$. A pattern can be considered as part of an array of zeroes and ones indexed by $\mathbb{Z}^2$ once the position of one of its elements is fixed: this is done by saying that the boldface 1 has position $(0, 0)$; This defines the first component u. The second component α is the position of the star.

4. The FCM of the Fibonacci sequence

Here σ stands for the Fibonacci substitution ($\sigma = (ab, a)$). Its unique fixed point F_σ is the Fibonacci sequence. It is well known that F_σ is special sturmian.

From Theorem 3.4, it is easy to get the corresponding FCM (see Fig. 1). In this section we study the sequence of sums of elements of rows and prove that it is image of a fixed point of a substitution. The same results holds for columns.

Set $[a, b] = a^{-1}b^{-1}ab$ and, for $n \geqslant 0$, $F_n = \sigma^n(a)$. We have $\sigma([a, b]) = [b, a]$ and $F_0F_1 = F_1F_0[a, b] = F_2[a, b]$. Therefore $F_1F_2 = F_3[b, a]$, $F_2F_3 = F_4[a, b]$, and $F_3F_4 = F_5[b, a]$.

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	⋯
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 1. The FCM of the Fibonacci sequence.

We define a morphism $\phi : \{A, B\}^* \to \{a, b\}^*$: $A \mapsto F_4$, $B \mapsto F_3[b, a]$. More explicitly, $\phi : \{A, B\}^* \to \{a, b\}^*$: $A \mapsto abaababa$, $B \mapsto ababa$. Then, we have $\phi(AB) = F_4F_3[b, a] = F_5[b, a] = F_3F_4$. The extension of φ to $\{A, B\}^\omega$ will still be denoted by φ.

Let τ be the invertible substitution $\tau = (AAB, AB)$. We have $\phi(\tau(A)) = \phi(AAB) = F_4F_4F_3[b, a] = F_4F_3F_4 = F_6 = \sigma^2(\phi(A))$ and $\phi(\tau(B)) = \phi(AB) = F_4F_3[b, a] = F_5[b, a] = F_3F_4 = \sigma^2(F_1F_2) = \sigma^2(\phi(B))$. It follows by induction that $\phi \circ \tau^n = \sigma^{2n} \circ \phi$.

We have the following result.

Proposition 4.1. *With the above notations,* $\phi(F_\tau) = F_\sigma$.

Proof. We have $\phi(F_\tau) = \phi(\lim \tau^n(A)) = \lim \sigma^{2n}(\phi(A)) = F_\sigma$. $\square$

With the notations of the end of the preceding section, we define two patterns

$$\alpha = \overline{\phi(A)} = \begin{array}{ccccc} & \mathbf{1} & 1 & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & * \end{array} \quad \text{and} \quad \beta = \overline{\phi(B)} = \begin{array}{cccc} & \mathbf{1} & 1 & \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & * \end{array}.$$

It results that

$$\bar{c}\alpha\alpha\beta\alpha\alpha\beta\alpha\beta\alpha\alpha\beta\alpha\alpha\beta \cdots = \bar{c}\,\overline{F_\tau}$$

is the pattern formed by the 1's in the FCM of σ.

For our purpose of studying the sums of rows, this decomposition is not convenient. Indeed, when concatenating two patterns α or β, there are rows, and columns, whose elements come from both patterns. This is why we need to modify these patterns. The first two modifications are suitable for studying the sums of rows, the third one allows to study the sums of columns.

First modification

$$\alpha_1 = \begin{array}{ccccc} \mathbf{1} & 1 & 1 & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & * & \end{array} \quad \text{and} \quad \beta_1 = \begin{array}{cccc} \mathbf{1} & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & * & \end{array}.$$

It is easy to see that

$$\alpha_1\alpha_1\beta_1\alpha_1\alpha_1\beta_1\alpha_1\beta_1\alpha_1\alpha_1\beta_1\alpha_1\alpha_1\beta_1 \cdots$$

is the pattern of the 1's in the FCM deprived of its first row.

Second modification

$$\alpha_2 = \begin{array}{cccccc} \mathbf{1} & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & 1 \\ & & & * & & \end{array} \quad \text{and} \quad \beta_2 = \begin{array}{ccccc} \mathbf{1} & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \\ & & * & & \end{array}.$$

It is easy to see that

$$\alpha_2\alpha_2\beta_2\alpha_2\alpha_2\beta_2\alpha_2\beta_2\alpha_2\alpha_2\beta_2\alpha_2\alpha_2\beta_2\cdots$$

is the pattern of the 1's in the FCM, with the only difference that $t_{0,-1} = 1$.

Third modification

$$\alpha_3 = \begin{array}{cccc} \mathbf{1} & & & \\ 1 & & & \\ 1 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & * \\ & 1 & 1 & \\ & & 1 & \\ & & 1 & \end{array} \quad \text{and} \quad \beta_3 = \begin{array}{ccc} \mathbf{1} & & \\ 1 & & \\ 1 & 1 & \\ 1 & 1 & * \\ 1 & 1 & \\ & 1 & \\ & 1 & \end{array}.$$

It is easy to see that

$$\alpha_3\alpha_3\beta_3\alpha_3\alpha_3\beta_3\alpha_3\beta_3\alpha_3\alpha_3\beta_3\alpha_3\alpha_3\beta_3\cdots$$

is the pattern of the 1's in the FCM, with the only difference that $t_{-1,1} = 1$ and the first column has been removed.

Now we consider a new sequence $\Sigma = s_0s_1s_2s_3s_4\cdots$ defined as follows:

$$s_n = \delta_{0,n} + \sum_j t_{n,j},$$

where δ is the Kronecker symbol. Its first 50 entries are

$$\Sigma = 33334333343343333433334334333343343333433334334333\cdots .$$

Its results of the second modification that Σ is the image of the fixed point of the substitution τ under the morphism $(A \mapsto 3334,\ B \mapsto 334)$.

In the same way, the third modification shows that the sequence of sums of columns of the FCM is related to the image of F_τ under the morphism $(A \mapsto 655,\ B \mapsto 55)$ in the same manner as previously: remove the first column of the FCM, and add 1 to the number of 1's in the first column of the resulting matrix.

Remark 4.2. The sequence Σ can also be seen as a sequence over the alphabet $\{\widehat{A} = 33,\ \widehat{B} = 334\}$, in this case it is the fixed point of the invertible substitution $(\widehat{A}\widehat{B},\ \widehat{A}\widehat{B}\widehat{B})$. If we take the alphabet $\{\widetilde{A} = 33, \widetilde{B} = 4\}$, Σ will be a substitutive sequence of the invertible substitution $(\widetilde{A}\widetilde{A}\widetilde{B}, \widetilde{A}\widetilde{B})$. But Σ itself is not sturmian over the alphabet $\{3, 4\}$ (both 3333 and 4334 are factors), therefore it is not a fixed point of an invertible substitution. Moreover we have the following stronger result.

Theorem 4.3. *Σ is not a fixed point of any nontrivial substitution over the alphabet* $\{3, 4\}$.

Proof. We know that Σ is the image of the fixed point F_γ of the substitution $\gamma = (\widetilde{A}\widetilde{A}\widetilde{B}, \widetilde{A}\widetilde{B})$ under the morphism $\phi : \widetilde{A} \mapsto 33,\ \widetilde{B} \mapsto 4$. Then F_γ is locally isomorphic to the Fibonacci sequence over the alphabet $\{\widetilde{A}, \widetilde{B}\}$ and so is free of fourth powers (i.e., has no nontrivial factor of the form u^4). Assuming that η is a substitution over $\{3, 4\}$ with Σ as a fixed point, then $\eta(3)^4$ is a factor of Σ, thus $\eta(3)$ cannot be the image under ϕ of a factor of F_γ. On the other hand, $\eta(3)$ is a prefix of Σ, hence $4 \not\triangleright \eta(3)$, and $3 \triangleright \eta(3)$. It is easy to see that $33334 \triangleleft \eta(3)$. Then, since $\eta(33) \prec \Sigma$, we have $33333 \prec \Sigma$, a contradiction. □

5. Invertible substitutions, singular decompositions and FCMs

In the previous section, we saw that, in the Fibonacci case, the 1's in the FCM form an invertible substitutive sequence over an alphabet of appropriate patterns. A natural question is to ask whether this is still so for general invertible substitutive sequence.

As said at the end of Section 1, we only need to consider the substitution in $\langle \pi = (b, a), \sigma = (ab, a) \rangle$. In this case, all substitutive sequences are special sturmian. First, we study the singular decompositions of these sequences.

Recall that $\mathcal{G}_\sigma$ is the set of substitutions of the following type

$$\tau = \sigma^{n_p} \circ \pi \circ \sigma^{n_{p-1}} \circ \pi \cdots \sigma^{n_2} \circ \pi \circ \sigma^{n_1},$$

where $n_j \geqslant 1 (1 \leqslant j \leqslant p)$. Throughout this section, we only discuss this kind of substitutions. Furthermore, we will assume that $\sum_{i=1}^{p} n_i + (p-1)$ is odd. The other cases can be discussed in the same way.

In this case, we have the following facts, the proof of which can be found in [11],

$$ab \triangleright \tau^{2n+1}(a),\ ba \triangleright \tau^{2n}(a),\ ba \triangleright \tau^{2n+1}(b),\ ab \triangleright \tau^{2n}(b), \quad \text{for } n \geqslant 1 \tag{4}$$

and

$$F_\tau = \tau^\infty(a) = a^{k_1} b a^{k_2} b \cdots a^{k_n} b \cdots, \tag{5}$$

where, for $j \geqslant 1,\ k_j = k_1$ or $k_j = k_1 + 1$.

Let us set

$$m(\tau) = k_1 + 1. \tag{6}$$

Lemma 5.1. *Let* $\tau \in \mathcal{G}_\sigma$.
(i) *If* $|\tau(b)| = 1$, *then* $\tau = (a^n b, a)$ *for some* n;
(ii) *If* $|\tau(b)| \geqslant 2$, *then* $\tau(a) = \tau(b)u = uv$, *where* $u, v \in S^*$ *and* $v, \tau(b)$ *differ merely by their last two letters*.

Theorem 5.2. *Let* $\tau \in \mathcal{G}_\sigma$. *Then*
1. *the morphisms*

$$\gamma = \left(ba^{-1}, (ba^{-1})^{1-m(\tau)}a\right) \circ \tau^2 \circ \left(a^{m(\tau)-1}b, a^{m(\tau)}b)\right)$$

and

$$\phi_1 = \imath_a \circ \tau \circ \left(a^{m(\tau)-1}b, a^{m(\tau)}b\right)$$

are invertible substitutions,
2. *one has* $aF_\tau = \phi_1(F_\gamma)$.

Before proving this theorem, we need to analyze the singular decomposition of the fixed point of a substitution in $\mathcal{G}_\sigma$.

Definition 5.3. Let τ be a substitution. The word $w_n = \alpha\tau^n(a)\beta^{-1}$ is called the singular word of τ of order n, where α, β are letters, $\alpha \neq \beta$ and $\beta \triangleright \tau^n(a)$.

Likewise, we can define the singular words associated with $\tau^n(b)$ (i.e., $w_{n,b} = \beta\tau^n(b)\alpha^{-1}$) and both w_n and $w_{n,b}$ are palindromes (see [11]), but this last fact will not be needed.

As said previously, we can write

$$\tau^2(a) = \left(\prod_{1\leqslant j\leqslant m} a^{k_j} b\right) a$$

and

$$\tau^2(b) = \left(\prod_{1\leqslant j\leqslant l} a^{k_j} b\right)$$

with $1 \leqslant l \leqslant m$, $k_1 = m(\tau) - 1$, and, for $1 \leqslant j \leqslant m$, $k_j \in \{m(\tau), m(\tau) - 1\}$. Of course, by $\left(\prod_{1\leqslant j\leqslant l} a^{k_j} b\right)$ we mean $a^{k_1}ba^{k_2}b\cdots a^{k_l}b$.

Let k be a positive integer. We have

$$\begin{aligned}\tau^2(a^k b) &= \left(\left(\prod_{1\leqslant j\leqslant m} a^{k_j} b\right) a\right)^k \left(\prod_{1\leqslant j\leqslant l} a^{k_j} b\right)\\ &= \left(\prod_{1\leqslant j\leqslant m} a^{k_j} b\right)\left(\prod_{1\leqslant j\leqslant m} a^{k'_j} b\right)^{k-1}\left(\prod_{1\leqslant j\leqslant l} a^{k'_j} b\right),\end{aligned}$$

where $k'_1 = k_1 + 1 = m(\tau)$ and, for $j \geqslant 2$, $k'_j = k_j$.

Therefore

$$a\tau^{2n+1}(a^k b)a^{-1} = \left(\prod_{1\leqslant j\leqslant m} a\tau^{2n-1}(a^{k_j}b)a^{-1}\right) \times \left(\prod_{1\leqslant j\leqslant m} a\tau^{2n-1}(a^{k'_j}b)a^{-1}\right)^{k-1} \left(\prod_{1\leqslant j\leqslant l} a\tau^{2n-1}(a^{k'_j}b)a^{-1}\right).$$

As a is a suffix of $\tau^{2n-1}(b)$, $u_{2n-1}(k) = a\tau^{2n-1}(a^k b)a^{-1}$ is a word on the alphabet $\{a, b\}$. So, we have the following decomposition

$$u_{2n+1}(k) = \left(\prod_{1\leqslant j\leqslant m} u_{2n-1}(k_j)\right) \times \left(\prod_{1\leqslant j\leqslant m} u_{2n-1}(k'_j)\right)^{k-1} \left(\prod_{1\leqslant j\leqslant l} u_{2n-1}(k'_j)\right), \tag{7}$$

which involves only the two words $u_{2n-1}(m(\tau))$ and $u_{2n-1}(m(\tau)-1)$. The singular word w_{2n-1} is a prefix of both of these words.

For $n \geqslant 0$, we define the following substitutions

$$\phi_{2n+1} = (u_{2n+1}(m(\tau)-1), u_{2n+1}(m(\tau))).$$

These substitutions are invertible: indeed

$$\phi_{2n+1} = \imath_a \circ \tau^{2n+1} \circ (a, a^{m(\tau)-1}b) \circ (b, ab).$$

Then formula (7) says that $\phi_{2n-1}^{-1} \circ \phi_{2n+1}$ is a substitution γ which does not depend on n. We have $\gamma = (ba^{-1}, (ba^{-1})^{1-m(\tau)}a) \circ \tau^2 \circ (a^{m(\tau)-1}b, a^{m(\tau)}b))$. This substitution is also invertible.

In other words, $\gamma(a)$ and $\gamma(b)$ are obtained by replacing $u_1(m(\tau)-1)$ by a and $u_1(m(\tau))$ by b in the decompositions of $u_3(m(\tau)-1)$ and $u_3(m(\tau))$ given by formula (7).

Since $k_1 = m(\tau) - 1$, a is a prefix of $\gamma(a)$ as well as of $\gamma(b)$. So, γ has a unique fixed point F_γ.

By induction,

$$\phi_{2n+1} = \phi_1 \circ \gamma^n.$$

Proof of Theorem 5.2. Indeed, one has

$$\begin{aligned} aF_\tau &= \lim a\tau^{2n+1}(a^{m(\tau)-1}b)a^{-1} = \lim \phi_{2n+1}(a) \\ &= \lim \phi_1(\gamma^n(a)) = \phi_1(F_\gamma). \qquad \square \end{aligned}$$

Remark 5.4. We have just seen that aF_τ is the fixed point of an invertible substitution over the alphabet $\{u_1(m(\tau)-1), u_1(m(\tau))\}$ (with a slight abuse of notation). As said at the end of Section 3, this means that the FCM of τ is obtained (with a few mismatches on the top rows) by binding the patterns corresponding to $u_1(m(\tau)-1)$ and $u_1(m(\tau))$ according to the sequence F_γ.

In the same way, we could have shown that F_τ can be viewed as the fixed point of the same substitution this time acting on the alphabet consisting of the two symbols $\tau(a^{m(\tau)-1}b)$ and $\tau(a^{m(\tau)}b)$.

The point of Theorem 5.2 is that it gives a decomposition of the FCM into patterns which can be modified in such a way that it gives the structure of the sequence $\Sigma = \left(\sum_j t_{i,j}\right)_{i\geqslant 1}$. Indeed, we have the following result.

Theorem 5.5. *For all* $\tau \in \mathcal{G}_\sigma$, *the sequence* $\left(\sum_j t_{i,j}\right)_{i\geqslant 1}$ *is the image of* F_γ (*where* γ *is defined in Theorem* 5.2) *under a morphism of* $\{a, b\}^*$ *into* S^*, *where S is a finite subset of* $\mathbb{N}$.

Before proving this theorem, we wish to work out two examples.

Example 5.6. The Fibonacci substitution $\sigma = (ab, a)$.

In this case, we have $\phi_1 = (aab, aabab)$ and $\gamma = (ab, abb)$.
The corresponding patterns of 1's in the FCM are

$$
\begin{array}{ccc} & \mathbf{1} & \\ 1 & 1 & 1 \\ 1 & 1 & * \end{array}
\qquad\qquad
\begin{array}{cccc} & \mathbf{1} & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & * \end{array}.
$$

These patterns can be modified in the following way

$$
\begin{array}{ccc} \mathbf{1} & 1 & 1 \\ 1 & 1 & 1 \\ & * & \end{array}
\qquad\qquad
\begin{array}{cccc} \mathbf{1} & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & * & \end{array}.
$$

The binding of these last patterns according to the order determined by the fixed point $ababbababbabb\cdots$ of γ yields the FCM deprived of its first row. Therefore the sequence Σ is obtained by replacing *a*'s by 33 and *b*'s by 343 in F_γ.

Example 5.7. $\tau = \sigma \circ \pi \circ \sigma = (aab, a)$.

We have $\phi_1 = (aaabaab, aaabaabaab)$ and $\gamma = (aabab, aababab)$. The corresponding patterns are

$$
\begin{array}{cccc} & \mathbf{1} & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ & 1 & 1 & 1 \\ & 1 & 1 & * \end{array}
\qquad\qquad
\begin{array}{ccccc} & \mathbf{1} & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \\ & & 1 & 1 & * \end{array}.
$$

The modified patterns are

$$\begin{array}{cccc} \mathbf{1} & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & * & \end{array} \qquad \begin{array}{ccccc} \mathbf{1} & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & * & \end{array}.$$

The corresponding sequence Σ is obtained by replacing a's by 23333 and b's by 2333333 in F_γ.

Now we can explain how the modification of patterns works in the general case.

Proof of Theorem 5.5. We know that aa is a prefix and ab a suffix of both words $u_1(m(\tau)-1)$ and $u_1(m(\tau))$. This implies that the tops of both the corresponding patterns are $\begin{array}{ccc} & \mathbf{1} & \\ 1 & 1 & ? \end{array}$ while their bottoms are $\begin{array}{ccc} 1 & 1 & 1 \\ & 1 & * \end{array}$. So we can remove the boldface **1** from the top and replace the asterisk by 1. So the top and bottom lines of both the modified patterns are $\mathbf{1}\ 1\ ?$ and $\begin{array}{ccc} 1 & 1 & 1 \\ & * & \end{array}$.

Then the binding of these modified patterns according to F_γ yields the FCM of τ, deprived of its first row. But now each row of the FCM comes from only one new pattern. This proves Theorem 5.5. □

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