

A Vectorial Multifractal Formalism

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ABSTRACT. We develop a formalism to perform the multifractal analysis of vector valued functions of balls in a metric space.

Foreword

In 1973, when I was studying the Hausdorff dimension of Borel sets which carry Riesz products, I was not aware it would orient my scientific life. One day, my advisor, Jean-Pierre Kahane, told me “I just met my friend Benoît Mandelbrot and he told me of a problem which could interest you. Indeed, I suspect that it has some close connection with what you have just done on Riesz Products”. The problem was to validate Benoît’s conjecture on the Hausdorff dimension of Borel sets which carry the random measures he had just introduced to model turbulence. Jean-Pierre Kahane was right: I solved this problem. This is why Benoît invited me to spend one year as his postdoc at the T.J. Watson Research Center of IBM at Yorktown Heights. I benefited of almost daily long interviews with him. Discussion always began on scientific topics, but very quickly the conversation took an unexpected turn and the most eclectic subjects came into play. Benoît freely associated the thoughts which came into his mind. What still amazes me is that he never got lost: although he opened lots of parentheses, he was able to close them in the right order, sometimes more than an hour later. This has been a very stimulating stay and the start of a close connection with him and of a long lasting collaboration.

This is why I am very happy to dedicate this article to Benoît.

1. Introduction

The term multifractal and the notion of multifractality appeared in [15] although multifractal objects had been defined and considered before [24, 25]. Since then this has been an active field of research (for instance, see [1, 3, 4, 6, 7, 8, 9, 12, 16, 18, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30, 35, 36, 37, 39, 40]). In the beginning, the multifractal formalism was restricted to the study of measures and was dependent on a hierarchy of “boxes”. In other terms, it took place on a tree. Later on, Olsen [36] defined a grid-free setting.

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Also, it appeared that one can analyse more general objects than measures: functions [18, 19, 20, 21, 22] and sequences of Choquet capacities [23].

Multifractal analysis involves two functions of sets, a measure and the diameter. Indeed the former is analyzed with respect to the latter, but it was observed in [31, 32] that there is some duality between these two functions.

The present article emphasizes this duality by replacing the diameter by a more general function defined on balls in a metric space and analysing functions defined on balls which are more general than measures. Moreover, these functions upon analysis can be Banach valued. This leads to results on the simultaneous behavior of possibly infinitely many local Hölder exponents.

2. Generalized Hausdorff and packing measures and dimensions

We deal with a metric space (\mathbb{X}, d) . By $B(x, r)$, where $x \in \mathbb{X}$ and $r > 0$, we mean the ball $\{y \in \mathbb{X} \mid d(x, y) \leq r\}$. We assume that this metric space fulfills the Besicovitch covering property. This means that there exists an integer constant θ with the following property: if $\{B(x, r_x)\}_{x \in A}$ is a family of closed balls of \mathbb{X} whose radii are bounded, one can extract from this family θ packings which altogether cover A . Euclidean spaces and totally disconnected metric spaces are of this kind.

Definition 1. A valuation on the metric space (\mathbb{X}, d) is a real function ξ defined on the set of balls of \mathbb{X} , subject to the condition that it goes to $+\infty$ as the radius goes to 0. More precisely, if $\xi(x, r)$ stands for the value of ξ on the ball $B(x, r) = \{y \in \mathbb{X} \mid d(x, y) \leq r\}$, then, for all $x \in \mathbb{X}$, one has $\lim_{r \searrow 0} \xi(x, r) = +\infty$.

When such a valuation is given, one sets

$$\mathbb{X}_n = \{x \in \mathbb{X} \mid \xi(x, r) > 1 \text{ for } r \leq 1/n\}.$$

2.1. Hausdorff measures. We define Hausdorff-like measures. A few modifications of the usual definitions [17, 41] are needed for technical reasons. If $A \subset \mathbb{X}$, $t > 0$, and $\delta > 0$, one sets

$$(1) \quad \delta \bar{\mathcal{H}}_\xi^t(A) = \inf \left\{ \sum_j e^{-t\xi(x_j, r_j)} \mid A \subset \bigcup_j B(x_j, r_j), r_j \leq \delta, \text{ and } x_j \in A \right\}$$

and

$$(2) \quad \bar{\mathcal{H}}_\xi^t(A) = \lim_{\delta \searrow 0} \delta \bar{\mathcal{H}}_\xi^t(A).$$

It is plain to check that, as functions of A , these quantities are σ -subadditive. But, because the covers are made of balls centered on A , one cannot assert that $\bar{\mathcal{H}}_\xi^t$ is non-decreasing. This is why one considers

$$(3) \quad \tilde{\mathcal{H}}_\xi^t(A) = \sup_{F \subset A} \bar{\mathcal{H}}_\xi^t(F).$$

Now $\tilde{\mathcal{H}}_\xi^t$ is a metric outer measure. Nevertheless, one more step is needed because one cannot assert that, as a function of t , $\tilde{\mathcal{H}}_\xi^t(A)$ is non-decreasing. Nevertheless it is so if A is a subset of one of the \mathbb{X}_n . This is why one sets

$$\mathcal{H}_\xi^t(A) = \lim_{n \rightarrow \infty} \tilde{\mathcal{H}}_\xi^t(A \cap \mathbb{X}_n).$$

This time, as a function of t , $\mathcal{H}_\xi^t(A)$ is non-increasing. More precisely, if $s < t$ and $\mathcal{H}_\xi^s(A) < +\infty$, then $\mathcal{H}_\xi^t(A) = 0$. This implies

$$\inf \{t \geq 0 \mid \mathcal{H}_\xi^t(A) = 0\} = \sup \{t \geq 0 \mid \mathcal{H}_\xi^t(A) = +\infty\}.$$

The common value of these bounds will be denoted by $\dim_\xi A$.

2.2. Packing measures. We define generalized packing measures and dimension along the same lines as in [42]. We set

$$\delta \overline{\mathcal{P}}_\xi^t(A) = \sup \sum_j e^{-t\xi(x_j, r_j)},$$

where this supremum is taken on collections $\{(x_j, r_j)\}$ such that $r_j \leq \delta$ and $\{B(x_j, r_j)\}$ is a centered packing of A (when considering packings we shall always mean centered packings). Then, we define

$$\begin{aligned} \overline{\mathcal{P}}_\xi^t(A) &= \lim_{\delta \searrow 0} \delta \overline{\mathcal{P}}_\xi^t(A), \\ \tilde{\mathcal{P}}_\xi^t(A) &= \inf \left\{ \sum_j \overline{\mathcal{P}}_\xi^t(F_j) \mid A \subset \bigcup_j F_j \right\}, \end{aligned}$$

and

$$\mathcal{P}_\xi^t(A) = \lim_{n \rightarrow \infty} \tilde{\mathcal{P}}_\xi^t(A \cap \mathbb{X}_n).$$

The set functions $\tilde{\mathcal{P}}_\xi^t$ and \mathcal{P}_ξ^t are metric outer measures and $\mathcal{P}_\xi^t(A)$ has the following property as a function of t : if $s < t$ and $\mathcal{P}_\xi^s(A) < +\infty$, then $\mathcal{P}_\xi^t(A) = 0$. This allows one to set

$$\text{Dim}_\xi A = \inf \{t \geq 0 \mid \mathcal{P}_\xi^t(A) = 0\} = \sup \{t \geq 0 \mid \mathcal{P}_\xi^t(A) = \infty\}.$$

LEMMA 1. *One has $\mathcal{H}_\xi^t \leq \theta \mathcal{P}_\xi^t$.*

Proof. Consider a positive number δ and a subset A of \mathbb{X} . One can extract θ packings from the family $\{B(x, \delta)\}_{x \in A}$ which altogether form a cover of A . This shows that $\delta \overline{\mathcal{H}}_\xi^t(A) \leq \theta \delta \overline{\mathcal{P}}_\xi^t(A)$ and $\overline{\mathcal{H}}_\xi^t(A) \leq \theta \overline{\mathcal{P}}_\xi^t(A)$.

If $A \subset \bigcup_{j \geq 0} A_j$, one has $\overline{\mathcal{H}}_\xi^t(A) \leq \sum_{j \geq 0} \overline{\mathcal{H}}_\xi^t(A_j) \leq \theta \sum_{j \geq 0} \overline{\mathcal{P}}_\xi^t(A_j)$, which implies $\overline{\mathcal{H}}_\xi^t(A) \leq \theta \tilde{\mathcal{P}}_\xi^t(A)$ for all $A \subset \mathbb{X}$, which in its turn implies $\tilde{\mathcal{H}}_\xi^t(A) \leq \theta \tilde{\mathcal{P}}_\xi^t(A)$.

COROLLARY 2. *One has $\dim_\xi \leq \text{Dim}_\xi$.*

REMARK. The usual choice for $\xi(x, r)$ is $-\log r$. In this case \dim_ξ and Dim_ξ are the ordinary Hausdorff and packing dimensions. But one could think of other choices. For instance, $\xi(x, r) = \log(1 + \log^+ 1/r)$ would allow one to distinguish among sets of zero dimension. In some other circumstances, for instance when \mathbb{X} is a Riemannian manifold, it could be convenient to take for $\xi(x, r)$ minus the logarithm of the volume of the ball $B(x, r)$.

2.3. The Minkowski-Bouligand dimension. One also defines an index Δ_ξ which generalizes the Minkowski-Bouligand dimension: for a bounded set A , one sets

$$\Delta_\xi(A) = \inf \left\{ t \geq 0 \mid \lim_{n \rightarrow +\infty} \overline{\mathcal{P}}_\xi^t(A \cap \mathbb{X}_n) = 0 \right\}.$$

It is easily checked that $\mathcal{P}_\xi^t(A) \leq \lim_{n \rightarrow +\infty} \overline{\mathcal{P}}_\xi^t(A \cap \mathbb{X}_n)$. Thus $\text{Dim}_\xi \leq \Delta_\xi$.

If A is unbounded, one chooses $x_0 \in \mathbb{X}$ and set

$$\Delta_\xi(A) = \lim_{n \rightarrow +\infty} \Delta_\xi(A \cap B(x_0, n)).$$

Due to the monotonicity of Δ_ξ , this definition does not depend on the choice of x_0 .

Now, we wish to give an alternate definition of Δ_ξ .

Definition 2. The valuation ξ is said to be normal if, for all $n \geq 1$ and all $\varepsilon > 0$, there exists $\rho > 0$, such that $\sum_{j \geq 0} e^{-\varepsilon \tilde{\xi}_n(\rho 2^{-j})} < +\infty$, where

$$\tilde{\xi}_n(t) = \inf_{x \in \mathbb{X}_n} \inf_{t/2 \leq r < t} \xi(x, r).$$

If A is a bounded subset of \mathbb{X} , we define

$$(4) \quad \delta \mathcal{P}_\xi^{*t}(A) = \sup \sum e^{-t\xi(x_j, r_j)},$$

where the supremum is taken over the collection of δ -packings $\{B(x_j, r_j)\}$ of A such that $\delta/2 < r_j \leq \delta$ and

$$(5) \quad \mathcal{P}_\xi^{*t}(A) = \sup_{n \geq 1} \limsup_{\delta \searrow 0} \delta \mathcal{P}_\xi^{*t}(A \cap \mathbb{X}_n).$$

Then we define

$$\Delta_\xi^*(A) = \inf \{ t \geq 0 \mid \mathcal{P}_\xi^{*t}(A) = 0 \}.$$

If A is unbounded, as previously we choose $x_0 \in \mathbb{X}$ and set

$$\Delta_\xi^*(A) = \lim_{n \rightarrow +\infty} \Delta_\xi^*(A \cap B(x_0, n)).$$

Then we have the following result.

PROPOSITION 3. *If the valuation ξ is normal, then $\Delta_\xi(A) = \Delta_\xi^*(A)$.*

Proof. We may suppose that A is bounded. This proof follows the same lines as in [42].

Since, for all $n \geq 1$, $\mathcal{P}_\xi^{*t}(A \cap \mathbb{X}_n) \leq \overline{\mathcal{P}}_\xi^t(A \cap \mathbb{X}_n)$, one has $\Delta_\xi^*(A) \leq \Delta_\xi(A)$.

We suppose that $\Delta_\xi(A) > 0$, otherwise we are finished. Let t and ε be two positive numbers such that $0 < t - \varepsilon < t < \Delta_\xi(A)$. Therefore, there exists $n \geq 1$ such that $\overline{\mathcal{P}}_\xi^t(A \cap \mathbb{X}_n) = +\infty$.

We define recursively a sequence $\{\eta_m\}_{m \geq 0}$. First, $\eta_0 = 2^{-k_0} \rho$, where ρ is given by the normality of ξ and k_0 is chosen so that $\eta_0 \leq 1/n$. Suppose η_m has been defined. Then there exists a $(\eta_m/2)$ -packing $\{B(x_j, r_j)\}$ of $A \cap \mathbb{X}_n$ with $\sum e^{-t\xi(x_j, r_j)} > 1$. There exists a positive integer $k \geq 1$ such that

$$\sum_{j: 2^{-(k+1)}\eta_m < r_j \leq 2^{-k}\eta_m} e^{-t\xi(x_j, r_j)} > e^{-\varepsilon \tilde{\xi}(2^{-k}\eta_m)} / \sum_{l \geq 1} e^{-\varepsilon \tilde{\xi}(2^{-l}\eta_m)}.$$

Then we set $\eta_{m+1} = 2^{-k}\eta_m$. It results that

$$\eta_{m+1} \mathcal{P}_\xi^{*t-\varepsilon}(A \cap \mathbb{X}_n) \geq \sum_{j:\eta_{m+1}/2 < r_j \leq \eta_{m+1}} e^{-t\xi(x_j, r_j)} e^{\varepsilon\xi(x_j, r_j)} > 1 / \sum_{l \geq 1} e^{-\varepsilon\tilde{\xi}(2^{-l}\eta_m)}.$$

Therefore $\mathcal{P}_\xi^{*t-\varepsilon}(A \cap \mathbb{X}_n) = +\infty$ and $\Delta_\xi^*(A \cap \mathbb{X}_n) \geq t - \varepsilon$. Due to the choice of t and ε , this means that $\Delta_\xi(A) \leq \Delta_\xi^*(A)$.

3. A multifractal formalism

We are given a valuation function ξ , and a function \varkappa from $\mathbb{X} \times \mathbb{R}^+$ to \mathbb{E}' , the dual of a separable real Banach space \mathbb{E} . We denote by $\langle \cdot, \cdot \rangle$ the duality bracket between \mathbb{E} and \mathbb{E}' .

We are going to define several quantities and sets, as in [36].

3.1. Multifractal Hausdorff and packing measures. For $A \subset \mathbb{X}$, $q \in \mathbb{E}$, $t \in \mathbb{R}$, and $\delta > 0$, we set

$$\delta \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A) = \sup \sum_j e^{-\langle q, \varkappa(x_j, r_j) \rangle + t\xi(x_j, r_j)},$$

where this supremum is taken on collections $\{(x_j, r_j)\}$ such that $r_j \leq \delta$ and $\{B(x_j, r_j)\}$ is a centered packing of A .

Then, as previously, we set

$$\begin{aligned} \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A) &= \lim_{\delta \searrow 0} \delta \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A), \\ \widetilde{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A) &= \inf \left\{ \sum_j \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(F_j) \mid A \subset \bigcup_j F_j \right\}, \end{aligned}$$

and

$$\mathcal{P}_{\xi, \varkappa}^{q, t}(A) = \lim_{n \rightarrow \infty} \widetilde{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A \cap \mathbb{X}_n).$$

Similarly, Hausdorff-like quantities are defined:

$$\delta \overline{\mathcal{H}}_{\xi, \varkappa}^{q, t}(A) = \inf \sum_j e^{-\langle q, \varkappa(x_j, r_j) \rangle + t\xi(x_j, r_j)},$$

where the infimum is taken over the families $\{(x_j, r_j)\}$ such that $\{B(x_j, r_j)\}$ is a centered δ -cover of A ,

$$\begin{aligned} \overline{\mathcal{H}}_{\xi, \varkappa}^{q, t}(A) &= \lim_{\delta \searrow 0} \delta \overline{\mathcal{H}}_{\xi, \varkappa}^{q, t}(A), \\ \widetilde{\mathcal{H}}_{\xi, \varkappa}^{q, t}(A) &= \sup_{F \subset A} \overline{\mathcal{H}}_{\xi, \varkappa}^{q, t}(F), \end{aligned}$$

and

$$\mathcal{H}_{\xi, \varkappa}^{q, t}(A) = \lim_{n \rightarrow \infty} \widetilde{\mathcal{H}}_{\xi, \varkappa}^{q, t}(A \cap \mathbb{X}_n).$$

REMARK. Of course, when $\varkappa = 0$, these measures reduce to the ones defined in Section 2.

These constructions have some similarities with the ones in Pesin's book [38]. Indeed, functions $\langle q, \varkappa \rangle$ and ξ would correspond in Pesin's terminology to $-\log \xi$ and $-\log \eta$, Pesin's function ψ would be the diameter and \mathcal{F} the collection of balls of \mathcal{X} . But there are two main differences. The first one is that we use centered covers and packings and the second is that we do not assume any continuity property of η with respect to ψ (in Pesin's notation). This is why two more steps are needed to construct these measures. Moreover, taking logarithms is not a mere rewriting; It allows to consider Banach valued functions \varkappa .

One defines, as in [36],

$$B_{\xi, \varkappa}(q) = \inf\{t \in \mathbb{R} \mid \mathcal{P}_{\xi, \varkappa}^{q, t}(\mathbb{X}) = 0\},$$

and

$$b_{\xi, \varkappa}(q) = \inf\{t \in \mathbb{R} \mid \mathcal{H}_{\xi, \varkappa}^{q, t}(\mathbb{X}) = 0\}.$$

As in the previous section, we have the inequality $b_{\xi, \varkappa} \leq B_{\xi, \varkappa}$.

PROPOSITION 4. *The function $B_{\xi, \varkappa}$ is convex.*

PROOF. The proof follows [36]. Let p and q be two points in \mathbb{E} ; take $t > B_{\xi, \varkappa}(p)$ and $u > B_{\xi, \varkappa}(q)$. So, for all $n \geq 1$, $\tilde{\mathcal{P}}_{\xi, \varkappa}^{p, t}(\mathbb{X}_n) = \tilde{\mathcal{P}}_{\xi, \varkappa}^{q, u}(\mathbb{X}_n) = 0$. Therefore, for each n one can find decompositions $\mathbb{X}_n = \bigcup_{j \geq 1} A_j = \bigcup_{k \geq 1} F_k$ such that $\sum_{j \geq 1} \overline{\mathcal{P}}_{\xi, \varkappa}^{p, t}(A_j) \leq 1$ and $\sum_{k \geq 1} \overline{\mathcal{P}}_{\xi, \varkappa}^{q, u}(F_k) \leq 1$. Then the logarithmic convexity of $(q, t) \mapsto \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A)$ implies

$$\overline{\mathcal{P}}_{\xi, \varkappa}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)u}(A_j \cap F_k) \leq \left(\overline{\mathcal{P}}_{\xi, \varkappa}^{p, t}(A_j \cap F_k)\right)^\alpha \left(\overline{\mathcal{P}}_{\xi, \varkappa}^{q, u}(A_j \cap F_k)\right)^{1-\alpha}$$

for all $\alpha \in (0, 1)$. Then, due to the Hölder inequality, one has

$$\begin{aligned} & \sum_{1 \leq j, k \leq m} \overline{\mathcal{P}}_{\xi, \varkappa}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)u}(A_j \cap F_k) \\ & \leq \left(\sum_{1 \leq j, k \leq m} \overline{\mathcal{P}}_{\xi, \varkappa}^{p, t}(A_j \cap F_k) \right)^\alpha \left(\sum_{1 \leq j, k \leq m} \overline{\mathcal{P}}_{\xi, \varkappa}^{q, u}(A_j \cap F_k) \right)^{1-\alpha} \\ & \leq \left(m \sum_{1 \leq j \leq m} \overline{\mathcal{P}}_{\xi, \varkappa}^{p, t}(A_j) \right)^\alpha \left(m \sum_{1 \leq k \leq m} \overline{\mathcal{P}}_{\xi, \varkappa}^{q, u}(F_k) \right)^{1-\alpha} \leq m. \end{aligned}$$

It results that

$$\begin{aligned} \mathcal{P}_{\xi, \varkappa}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)u} \left(\bigcup_{1 \leq j, k < m} A_j \cap F_k \right) &= \\ \tilde{\mathcal{P}}_{\xi, \varkappa}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)u} \left(\bigcup_{1 \leq j, k < m} A_j \cap F_k \right) &\leq m. \end{aligned}$$

Therefore, if $\varepsilon > 0$, one has $\mathcal{P}_{\xi, \varkappa}^{\alpha p + (1-\alpha)q, \alpha t + (1-\alpha)u + \varepsilon}(\mathbb{X}_n) = 0$. The proposition then follows.

3.2. Local Hölder exponent – Chernoff-like inequalities. If $\alpha \in \mathbb{E}'$ and $E \subset \mathbb{E}$, we set

$$(6) \quad X_{\xi, \varkappa}(\alpha, E) = \left\{ x \mid \limsup_{r \searrow 0} \frac{\langle w, \varkappa(x, r) \rangle}{\xi(x, r)} \leq \langle w, \alpha \rangle \text{ for all } w \in E \right\}.$$

The set $X_{\xi, \varkappa}(\alpha, \mathbb{E})$ will simply be denoted by $X_{\xi, \varkappa}(\alpha)$. This is the set of points x such that $\lim_{r \searrow 0} \frac{\varkappa(x, r)}{\xi(x, r)} = \alpha$ (according to the $\sigma(\mathbb{E}, \mathbb{E}')$ topology).

PROPOSITION 5. *One has*

$$\text{Dim}_{\xi} X_{\xi, \varkappa}(\alpha, \{q\}) \leq \langle q, \alpha \rangle + B_{\xi, \varkappa}(q).$$

Proof. Let ε and η be two positive numbers, $q \in \mathbb{E}$, and m and n two positive integers such that $m \geq n$.

$$\text{Set } A_{m, n}(\varepsilon) = \left\{ x \in \mathbb{X}_n \mid \frac{\langle q, \varkappa(x, r) \rangle}{\xi(x, r)} \leq \langle q, \alpha \rangle + \varepsilon \text{ for } r < 1/m \right\}.$$

Then, if $\{B(x_j, r_j)\}$ is a δ -packing of a subset F of $A_{m, n}(\varepsilon)$, with $\delta < 1/m$, one has

$$\sum_j e^{-\left(\langle q, \alpha \rangle + \varepsilon + B_{\xi, \varkappa}(q) + \eta\right) \xi(x_j, r_j)} \leq \sum_j e^{-\left(\langle q, \varkappa(x_j, r_j) \rangle + \xi(x_j, r_j)(B_{\xi, \varkappa}(q) + \eta)\right)},$$

from which it follows

$$\overline{\mathcal{P}}_{\xi}^{\langle q, \alpha \rangle + \varepsilon + B_{\xi, \varkappa}(q) + \eta}(F) \leq \overline{\mathcal{P}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q) + \eta}(F).$$

Since $\mathcal{P}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q) + \eta}(\mathbb{X}) = 0$, it follows that, for all n , $\widetilde{\mathcal{P}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q) + \eta}(\mathbb{X}_n) = 0$. Therefore, $\inf \left\{ \sum_j \overline{\mathcal{P}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q) + \eta}(F_j) \mid \mathbb{X}_n \subset \bigcup F_j \right\} = 0$. It results

$$\mathcal{P}_{\xi}^{\langle q, \alpha \rangle + \varepsilon + B_{\xi, \varkappa}(q) + \eta}(A_{m, n}(\varepsilon)) = \widetilde{\mathcal{P}}_{\xi}^{\langle q, \alpha \rangle + \varepsilon + B_{\xi, \varkappa}(q) + \eta}(A_{m, n}(\varepsilon)) = 0.$$

So, we have $\text{Dim}_{\xi} A_{m, n}(\varepsilon) \leq \langle q, \alpha \rangle + \varepsilon + B_{\xi, \varkappa}(q)$. But as

$$\left\{ x \in \mathbb{X} \mid \limsup_{r \searrow 0} \frac{\langle q, \varkappa(x, r) \rangle}{\xi(x, r)} \leq \langle q, \alpha \rangle \right\} \subset \bigcup_{n \geq 1} \bigcap_{p \geq 1} \bigcup_{m \geq n} A_{m, n}(1/p),$$

we get the announced inequality.

COROLLARY 6. *For $\alpha \in \mathbb{E}'$ and $E \subset \mathbb{E}$, one has*

$$\text{Dim}_{\xi} X_{\xi, \varkappa}(\alpha, E) \leq \inf_{q \in E} \langle q, \alpha \rangle + B_{\xi, \varkappa}(q).$$

REMARK. Of course, if this formula gives a negative dimension, this means that the corresponding set is empty.

PROPOSITION 7. *Let $X_{\xi, \varkappa}^*(\alpha, E)$ be the set defined as in Equation (6), but with \liminf instead of \limsup . Then*

$$\dim_{\xi} X_{\xi, \varkappa}^*(\alpha, E) \leq \inf_{q \in E} \langle q, \alpha \rangle + B_{\xi, \varkappa}(q).$$

Proof. As above, it is enough to consider the case $E = \{q\}$. This time, one sets

$$A_{m,n} = \left\{ x \in \mathbb{X}_n \mid \liminf_{r \searrow 0} \frac{\langle q, \varkappa(x, r) \rangle}{\xi(x, r)} \leq \langle q, \alpha \rangle + \frac{1}{m} \right\}$$

Let F be a subset of $A_{m,n}$ and δ a positive number less than $1/n$. For any $x \in F$, there exists $r_x \leq \delta$ such that $\langle q, \varkappa(x, r_x) \rangle \leq (\langle q, \alpha \rangle + \frac{1}{m}) \xi(x, r_x)$. Since the Besicovitch covering property is assumed to hold, one can extract from the family $\{B(x, r_x)\}_{x \in F}$ a family $\{B_{i,j}\}_{1 \leq i \leq \theta, j \geq 0}$ of balls which covers F and such that, for every i , $\{B_{i,j}\}_{j \geq 0}$ is a packing. Then, for any $\eta > 0$, one gets

$$\delta \overline{\mathcal{H}}_{\xi}^{\langle q, \alpha \rangle + 1/m + B_{\xi, \varkappa}(q) + \eta}(F) \leq \theta \delta \overline{\mathcal{P}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q) + \eta}(F).$$

It is then easy to conclude that $\mathcal{H}_{\xi}^{\langle q, \alpha \rangle + 1/m + B_{\xi, \varkappa}(q) + \eta}(A_{m,n}) = 0$. This implies $\dim_{\xi} A_{m,n} \leq \langle q, \alpha \rangle + 1/m + B_{\xi, \varkappa}(q)$. The proposition follows from $X_{\xi, \varkappa}^*(\alpha, q) = \bigcup_{n \geq 1} \bigcap_{m \geq 1} A_{m,n}$.

3.3. The converse inequality. We will use the following notations:

- If $|B_{\xi, \varkappa}(q)| < \infty$ and $v \in \mathbb{E}$, one sets

$$\partial_v B_{\xi, \varkappa}(q) = \lim_{t \searrow 0} \frac{B_{\xi, \varkappa}(q + tv) - B_{\xi, \varkappa}(q)}{t};$$

- $B'_{\xi, \varkappa}(q)$ stands for the derivative (considered as an element of \mathbb{E}') of $B_{\xi, \varkappa}$ at point q , when it exists.

When $B_{\xi, \varkappa}$ has a partial derivative at point q along the direction v , one has $\partial_{-v} B_{\xi, \varkappa}(q) = -\partial_v B_{\xi, \varkappa}(q)$. When $B'_{\xi, \varkappa}(q)$ exists, $\partial_v B_{\xi, \varkappa}(q) = \langle v, B'_{\xi, \varkappa}(q) \rangle$.

LEMMA 8. *Let $v \in \mathbb{E}$ and q such that $|B_{\xi, \varkappa}(q)| < \infty$. Then*

$$\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)} \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} < -\partial_v B_{\xi, \varkappa}(q) \right\} = 0.$$

Proof. We follow [4]. Take $\lambda > \partial_v B_{\xi, \varkappa}(q)$ and choose $t > 0$ such that $B_{\xi, \varkappa}(q + tv) < B_{\xi, \varkappa}(q) + \lambda t$. Consider the set

$$F_n = \left\{ x \in \mathbb{X}_n \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} < -\lambda \right\}.$$

Given any $\delta \in (0, 1/n)$, for each $x \in F_n$, one can find $r_x > 0$ such that $r_x < \delta$ and $\langle v, \varkappa(x, r_x) \rangle + \lambda \xi(x, r_x) \leq 0$. Let F' be any non-empty subset of F_n . Due to the Besicovitch covering property of \mathbb{X} , one can find θ sequences $(x_{i,j})_j$ ($1 \leq i \leq \theta$) of points of F' such that, for $i = 1, 2, \dots, \theta$, the balls $(B(x_{i,j}, r_{x_{i,j}}))_j$ form a packing of F' and that these packings altogether form a cover of F' . Then one has

$$\begin{aligned} \delta \overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F') &\leq \sum_{1 \leq i \leq \theta} \sum_j e^{-\left(\langle q, \varkappa(x_{i,j}, r_{x_{i,j}}) \rangle + B_{\xi, \varkappa}(q) \xi(x_{i,j}, r_{x_{i,j}})\right)} \\ &\leq \sum_{1 \leq i \leq \theta} \sum_j e^{-\left(\langle q + tv, \varkappa(x_{i,j}, r_{x_{i,j}}) \rangle + (B_{\xi, \varkappa}(q) + \lambda t) \xi(x_{i,j}, r_{x_{i,j}})\right)} \\ &\leq \theta \delta \overline{\mathcal{P}}_{\varkappa, \xi}^{q + tv, B_{\xi, \varkappa}(q) + \lambda t}(F'). \end{aligned}$$

By letting δ go to 0, one obtains

$$(7) \quad \overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F') \leq \theta \overline{\mathcal{P}}_{\xi, \varkappa}^{q + tv, B_{\xi, \varkappa}(q) + \lambda t}(F').$$

Now, if $F' = \bigcup F'_j$ is written as a countable union, by (7), one gets

$$\overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F') \leq \sum \overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F'_j) \leq \theta \sum \overline{\mathcal{P}}_{\xi, \varkappa}^{q+tv, B_{\xi, \varkappa}(q)+\lambda t}(F'_j).$$

So, by the definitions of \mathcal{H} and \mathcal{P} , one gets

$$\overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F') \leq \theta \mathcal{P}_{\xi, \varkappa}^{q+tv, B_{\xi, \varkappa}(q)+\lambda t}(F') = 0,$$

and, finally,

$$\widetilde{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F_n) = 0.$$

LEMMA 9. *Let $x \in \mathbb{X}$. Consider the function $\rho_x(v) = \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)}$ and the cone $C_x = \{v \in \mathbb{E} \mid \rho_x(v) > -\infty\}$. The function ρ_x is concave and the cone C_x is convex. If the interior C_x° of C_x is nonempty two alternatives may occur: either $\rho_x(v) = +\infty$ for one $v \in C_x^\circ$ and then $\rho_x(v) = +\infty$ for all $v \in C_x^\circ$, or ρ_x is continuous on C_x° .*

Proof. Suppose $C_x^\circ \neq \emptyset$ and $\rho_x(v) < \infty$ for all $v \in C_x^\circ$. For all $\varepsilon > 0$ set

$$\rho_{x, \varepsilon} = \inf_{r < \varepsilon} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)}.$$

Given two positive integers m and n , consider the set $F_{m, n} = \left\{v \in C_x^\circ \mid \rho_{x, \frac{1}{m}} \geq -n\right\}$.

These sets are closed in C_x° because the functions $\rho_{x, \frac{1}{m}}$ are upper semi-continuous and one has $C_x^\circ = \bigcup_{m, n > 0} F_{m, n}$. Therefore, due to Baire's theorem, one of these sets

has a nonempty interior. As $\rho_x \geq \rho_{x, \varepsilon}$, this means that C_x° contains an open ball on which ρ_x is bounded from below. This implies that the concave function ρ_x is locally bounded on C_x° and therefore is continuous on C_x° .

PROPOSITION 10. *If $|B_{\xi, \varkappa}(q)| < \infty$ and if the function $v \mapsto \partial_v B_{\xi, \varkappa}(q)$ is lower semi-continuous, one has*

$$\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)} \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} < -\partial_v B_{\xi, \varkappa}(q) \text{ for some } v \in \mathbb{E} \right\} = 0.$$

Proof. Let E be a dense countable subset of \mathbb{E} . It results from the properties of ρ_x and the semi-continuity of $\partial_v B_{\xi, \varkappa}(q)$, that

$$\left\{ x \mid \exists v \in \mathbb{E}, \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} < -\partial_v B_{\xi, \varkappa}(q) \right\} \subset \left\{ x \mid \exists v \in E, \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} < -\partial_v B_{\xi, \varkappa}(q) \right\}.$$

One concludes by using the preceding lemma.

Of course this proposition has some interest only if $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) > 0$. If it is so, then for almost every x with respect to $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}$, one has

$$(8) \quad \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} \geq -\partial_v B_{\xi, \varkappa}(q)$$

for all $v \in \mathbb{E}$.

By changing v into $-v$ in Formula (8), one gets

$$\limsup_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} \leq -\partial_v B_{\xi, \varkappa}(q).$$

PROPOSITION 11. *If, for some q , $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) > 0$, and if the function $v \mapsto \partial_v B_{\xi, \varkappa}(q)$ is lower semi-continuous, then*

$$\dim_{\xi} \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} + \partial_v B_{\xi, \varkappa}(q) \geq 0 \text{ for all } v \in \mathbb{E} \right\} \geq B_{\xi, \varkappa}(q) - \partial_q B_{\xi, \varkappa}(q).$$

Proof. Set $X = \left\{ x \mid \liminf_{r \searrow 0} \frac{\langle v, \varkappa(x, r) \rangle}{\xi(x, r)} + \partial_v B_{\xi, \varkappa}(q) \geq 0 \text{ for all } v \in \mathbb{E} \right\}$.

It results from the preceding lemma that there exists n such that

$$\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(X \cap \mathbb{X}_n) > 0.$$

Take $\varepsilon > 0$. For $m \geq n$, consider

$$F_{m, \varepsilon} = \{x \in X \cap \mathbb{X}_n \mid \langle q, \varkappa(x, r) \rangle + (\partial_q B_{\xi, \varkappa}(q) + \varepsilon) \xi(x, r) > 0 \text{ for } r \leq 1/m\}.$$

As $X \cap \mathbb{X}_n = \bigcup_{m \geq 1} F_{m, \varepsilon}$, there exists m such that $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F_{m, \varepsilon}) > 0$. Therefore, there exist n and a subset F of $F_{m, \varepsilon}$ such that $\overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F) > 0$. If $\{B(x_j, r_j)\}$ is a centered δ -cover of F , with $\delta < \min\{1/n, 1/m\}$, one has

$$\begin{aligned} \sum e^{-(B_{\xi, \varkappa}(q) - \partial_q B_{\xi, \varkappa}(q) - \varepsilon) \xi(x_j, r_j)} &\geq \sum e^{-(\langle q, \varkappa(x_j, r_j) \rangle + B_{\xi, \varkappa}(q) \xi(x_j, r_j))} \\ &\geq \delta \overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F), \end{aligned}$$

which gives

$$\mathcal{H}_{\xi}^{B_{\xi, \varkappa}(q) - \partial_q B_{\xi, \varkappa}(q) - \varepsilon}(F_{m, \varepsilon}) \geq \overline{\mathcal{H}}_{\xi}^{B_{\xi, \varkappa}(q) - \partial_q B_{\xi, \varkappa}(q) - \varepsilon}(F) \geq \overline{\mathcal{H}}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(F) > 0.$$

So, $\dim_{\xi} X \geq \dim_{\xi} F_{m, \varepsilon} \geq B_{\xi, \varkappa}(q) - \partial_q B_{\xi, \varkappa}(q) - \varepsilon$.

THEOREM 12. *If, for some q , the function $B_{\xi, \varkappa}$ is differentiable with derivative $B'_{\xi, \varkappa}(q)$ and if $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) > 0$, then one has $b_{\xi, \varkappa}(q) = B_{\xi, \varkappa}(q)$ and*

$$\dim_{\xi} X(-B'_{\xi, \varkappa}(q)) = \text{Dim}_{\xi} X(-B'_{\xi, \varkappa}(q)) = B_{\xi, \varkappa}^*(-B'_{\xi, \varkappa}(q)).$$

This results from Corollary 6 and Proposition 11.

3.4. Frostman and Gibbs measures. The hypothesis $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) > 0$ of Theorem 12 is usually not easily checked. In this respect, the following lemma may be useful.

LEMMA 13. *If there exists a measure $\mu^{[q]}$ such that for $\mu^{[q]}$ -almost every x one has $\limsup_{r \searrow 0} \frac{\mu^{[q]}(B(x, r))}{e^{-(\langle q, \varkappa(x, r) \rangle + B_{\xi, \varkappa}(q) \xi(x, r))}} < +\infty$, then $\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) > 0$.*

Proof. There exist two positive numbers C and η and a Borel subset A of \mathbb{X} such that $\mu^{[q]}(A) > 0$ and such that, for all $x \in A$ and $r \leq \eta$, one has $\mu^{[q]}(B(x, r)) \leq C e^{-(\langle q, \varkappa(x, r) \rangle + B_{\xi, \varkappa}(q) \xi(x, r))}$. Then

$$\mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(\mathbb{X}) \geq \mathcal{H}_{\xi, \varkappa}^{q, B_{\xi, \varkappa}(q)}(A) \geq \frac{\mu^{[q]}(A)}{C}.$$

We call such a measure a Frostman measure at q .

When there exists a Borel measure $\mu^{[q]}$, and two positive numbers η and C such that, for all $x \in \mathbb{X}$, and for all $r \leq \eta$, one has

$$(9) \quad \frac{1}{C} e^{-\langle (q, \varkappa(x, r) + B_{\xi, \varkappa}(q) \xi(x, r)) \rangle} \leq \mu^{[q]}(B(x, r)) \leq C e^{-\langle (q, \varkappa(x, r) + B_{\xi, \varkappa}(q) \xi(x, r)) \rangle}$$

we say that $\mu^{[q]}$ is a Gibbs measure at q .

For construction of Gibbs measures in the ultrametric context, see [33, 34].

3.5. The Λ and τ functions. One also defines an index $\Delta_{\xi, \varkappa}^q$ which generalizes the Minkowski-Bouligand dimension to the setting of multifractals: for a bounded set A , one sets

$$\Delta_{\xi, \varkappa}^q(A) = \inf \left\{ t \geq 0 \mid \lim_{n \rightarrow +\infty} \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A \cap \mathbb{X}_n) = 0 \right\}.$$

If A is unbounded, one chooses $x_0 \in \mathbb{X}$ and set

$$\Delta_{\xi, \varkappa}^q(A) = \lim_{n \rightarrow +\infty} \Delta_{\xi, \varkappa}^q(A \cap B(x_0, n)).$$

As in Section 2.3, this definition does not depend on the choice of x_0 .

Since the function $(q, t) \mapsto \overline{\mathcal{P}}_{\xi, \varkappa}^{q, t}(A)$ is log-convex, the function $q \mapsto \Delta_{\xi, \varkappa}^q(A)$ is convex.

Definition 3. We set $\Lambda_{\xi, \varkappa}(q) = \Delta_{\xi, \varkappa}^q(\mathbb{X})$.

Obviously, one has

$$B_{\xi, \varkappa}(q) \leq \Lambda_{\xi, \varkappa}(q).$$

If A is a bounded subset of \mathbb{X} , we define

$$(10) \quad \delta \mathcal{P}_{\xi, \varkappa}^{*q, t}(A) = \sup \sum e^{-\langle (q, \varkappa(x_j, r_j)) + t \xi(x_j, r_j) \rangle},$$

where the supremum is taken over the collection of δ -packings $\{B(x_j, r_j)\}$ of A such that $\delta/2 < r_j \leq \delta$ and

$$(11) \quad \mathcal{P}_{\xi, \varkappa}^{*t}(A) = \sup_{n \geq 1} \limsup_{\delta \searrow 0} \delta \mathcal{P}_{\xi, \varkappa}^{*t}(A \cap \mathbb{X}_n).$$

Then we define

$$\Delta_{\xi, \varkappa}^{*q}(A) = \inf \{ t \geq 0 \mid \mathcal{P}_{\xi, \varkappa}^{*q, t}(A) = 0 \}.$$

If A is unbounded, as previously we choose $x_0 \in \mathbb{X}$ and set

$$\Delta_{\xi, \varkappa}^{*q}(A) = \lim_{n \rightarrow +\infty} \Delta_{\xi, \varkappa}^{*q}(A \cap B(x_0, n)).$$

Definition 4. $\tau_{\xi, \varkappa}(q) = \Delta_{\xi, \varkappa}^{*q}(\mathbb{X})$.

Then we have the following result.

PROPOSITION 14. *If the valuation ξ is normal, then $\Delta_{\xi, \varkappa}^q(A) = \Delta_{\xi, \varkappa}^{*q}(A)$ and $\Lambda_{\xi, \varkappa} = \tau_{\xi, \varkappa}$.*

Proof. It follows the same lines as the one of Proposition 3.

4. Applications

We are given an integer $c \geq 2$. In this section, \mathbb{X} is $\{0, 1, 2, \dots, c-1\}^{\mathbb{N}}$ endowed with the usual ultrametric distance: two sequences $(\varepsilon_n)_{n \geq 0}$ and $(\alpha_n)_{n \geq 0}$ are distant from c^{-k} if $\varepsilon_k \neq \alpha_k$ and if $\varepsilon_j = \alpha_j$ for all j such that $0 \leq j < k$. We take $\xi(x, c^{-k}) = k \log c$.

Let $p = \{(p_{l,0}, p_{l,1}, \dots, p_{l,c-1})\}_{0 \leq l < \nu}$ be a family of positive numbers. If $x = (x_n)_{n \geq 0} \in \mathbb{X}$, one sets $\varkappa(x, c^{-k}) = (-\log \prod_{0 \leq j < k} p_{l, x_j})_{0 \leq l < \nu}$. In what follows, we drop the indices ξ and \varkappa and the space \mathbb{E}' is \mathbb{R}^ν .

It is easily seen that $\tau(q) = \log_c \left(\sum_{0 \leq j < c} \prod_{0 \leq l < \nu} p_{l,j}^{q_l} \right)$. As ξ clearly is normal, this also is the Λ function.

If $q \in \mathbb{R}^\nu$, one sets, for $1 \leq j \leq \nu$,

$$r_j = \prod_{0 \leq l < \nu} p_{l,j}^{q_l} / \sum_{0 \leq k < c} \prod_{0 \leq l < \nu} p_{l,k}^{q_l}.$$

A measure $\mu^{[q]}$ is defined on \mathbb{X} by the formula

$$\mu^{[q]}(B(x, c^{-k})) = \prod_{l=0}^{k-1} r_{x_l}.$$

It is easy to check that

$$\mu^{[q]}(B(x, c^{-k})) = e^{-\langle q, \varkappa(x, c^{-k}) \rangle + \tau(q) \xi(x, c^{-k})}.$$

This implies $\mathcal{H}^{q, \tau(q)}(\mathbb{X}) > 0$, which has two consequences: $b_{\xi, \varkappa}(q) = B_{\xi, \varkappa}(q) = \tau(q)$ and the fact that the multifractal formalism holds for all q .

It will be convenient to introduce some more notations. If $x = (x_n)_{n \geq 0} \in \mathbb{X}$, one sets

$$(12) \quad \varphi_n(x, j) = \frac{1}{n} \text{card}\{0 \leq k < n \mid x_k = j\}$$

for $j = 0, 1, \dots, c-1$.

With this notation, one has

$$(13) \quad \varkappa(x, c^{-k}) = \left(-k \sum_{0 \leq j < c} \varphi_k(x, j) \log p_{l,j} \right)_{0 \leq l < \nu}.$$

Now we are ready to prove a slight generalization of Eggleston's theorem [10]. As a matter of fact, this result can be obtained by Theorem 2 in [13]. See also [14] for a more general principle.

THEOREM 15. *Let $\nu < c$ and $f_0, f_1, \dots, f_{\nu-1}$ be positive numbers such that $\sum_{0 \leq j < \nu} f_j \leq 1$. Then, the Hausdorff dimension of the set*

$$\left\{ x \in \mathbb{X} \mid \lim_{n \rightarrow +\infty} \varphi_n(x, j) = f_j \text{ for } 0 \leq j < \nu \right\}$$

is equal to

$$- \left(1 - \sum_{0 \leq j < \nu} f_j \right) \log_c \frac{1 - \sum_{0 \leq j < \nu} f_j}{c - \nu} - \sum_{0 \leq j < \nu} f_j \log_c f_j$$

(of course $0 \log 0$ is interpreted to be 0).

Proof. Take $p_{j,j} = c^{-1}$ and $p_{l,j} = 1$ if $l \neq j$. Then

$$\tau(q) = \log_c \left(c - \nu + \sum_{0 \leq j < \nu} c^{-qj} \right)$$

and

$$\frac{\varkappa(x, c^{-k})}{\xi(x, c^{-k})} = (\varphi_k(x, j))_{0 \leq j < \nu}.$$

Then, it is easy to complete the computation of the Legendre transform.

It will prove convenient to set $H_c(x_0, x_1, \dots, x_{c-1}) = -\sum_{j=0}^{c-1} x_j \log_c x_j$.

Here is now a generalization of a theorem by Besicovitch [5].

THEOREM 16. *Suppose $\nu < c$. Let $f_0, f_1, \dots, f_{\nu-1}$ be non-negative numbers and consider the set*

$$B_f = \left\{ x \in \mathbb{X} \mid \limsup_{n \rightarrow \infty} \varphi_j(x, n) \leq f_j \text{ for } 0 \leq j \leq c-2 \right\}.$$

Let $f_0^* \geq f_1^* \geq \dots \geq f_{c-1}^*$ be the sequence $(f_j)_{0 \leq j < \nu}$ rearranged in decreasing order, and $f_j^{**} = \sum_{j \leq k < \nu} f_k^*$. Then we have

- (1) If $(c-\nu)f_0^* + f_0^{**} < 1$, then $\dim B_f = H_c(f_0^*, \dots, f_{\nu-1}^*, \frac{1-f_0^{**}}{c-\nu}, \frac{1-f_0^{**}}{c-\nu}, \dots)$.
- (2) For $0 \leq k < \nu-1$, if $(c-\nu+k)f_k^* + f_k^{**} \geq 1$ and $(c-\nu+k+1)f_{k+1}^* + f_{k+1}^{**} < 1$, then $\dim E = H_c(f_k^*, \dots, f_{\nu-1}^*, \frac{1-f_{k+1}^{**}}{c-\nu+k+1}, \frac{1-f_{k+1}^{**}}{c-\nu+k}, \dots)$.
- (3) If $f_{\nu-1}^* \geq \frac{1}{c}$, then $\dim E = 1$.

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References

- [1] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*. Math. Nachr. **181** (1996), 5–42.
- [2] J. Barral, F. Ben Nasr, and J. Peyrière, *Comparing multifractal formalisms: the neighboring boxes condition*. To appear in Asian J. of Math.
- [3] F. Ben Nasr, *Analyse multifractale de mesures*. Comptes Rendus Acad. Sc. Paris, Série I, **319** (1994), 807–810.
- [4] F. Ben Nasr, I. Bhourri, and Y. Heurteaux, *The validity of the multifractal formalism: results and examples*. Adv. in Math. **165** (2002), 264–284.
- [5] A.S. Besicovitch, *On the sum of digits of real numbers represented in the dyadic system*. Math. Ann. **110** (1934), 1–18.
- [6] I. Bhourri, *Une condition de validité du formalisme multifractal pour les mesures*. Thèse de Doctorat. Tunis, 1999.
- [7] G. Brown, G. Michon, and J. Peyrière, *On the multifractal analysis of measures*. J. Stat. Phys. **66** (1992), 775–790.
- [8] P. Collet, J. Lebowitz, and A. Porzio, *The dimension spectrum of some dynamical systems*. J. Stat. Phys. **47** (1987), 609–644.
- [9] G.A. Edgar and R.D. Mauldin, *Multifractal decomposition of digraph recursive fractals*. Proc. London Math. Soc. **65** (1992), 604–628.
- [10] H.G. Eggleston, *The fractional dimension of a set defined by decimal properties*. Q. J. Math. Oxford, Ser. **20** (1949), 31–46.
- [11] K.J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*. J. Wiley (New York), 1990.
- [12] K.J. Falconer, *The multifractal spectrum of statistically self-similar measures*. J. Theor. Prob. **7** (1994), 681–702.

- [13] A.-H. Fan, D.-J. Feng, *On the distribution of long-term time averages on symbolic space*. J. Stat. Phys., vol. **99**, Nos.3/4 (2000), 813–856.
- [14] A.-H. Fan, D.-J. Feng, and J. Wu, *Recurrence, dimension and entropy*. J. London Math. Soc., (2) **64** (2001), 229–244.
- [15] U. Frisch and G. Parisi, *Fully developed turbulence and intermittency in turbulence, and predictability in geophysical fluid dynamics and climate dynamics*. In: International School of Physics “Enrico Fermi”, course 88. M. Ghil, ed. (North-Holland, Amsterdam, 1985, p. 84).
- [16] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.J. Shraiman, *Fractal measures and their singularities: The characterization of strange sets*. Phys. Rev. A **33** (1986), 1141–1151.
- [17] F. Hausdorff, *Dimension und äusseres mass*. Math. Ann. **79** (1919), 157–179.
- [18] S. Jaffard, *Sur la dimension de Hausdorff des points singuliers d’une fonction*. Comptes Rendus Acad. Sc. Paris, Série I, **314** (1991), 31–36.
- [19] S. Jaffard, *Construction de fonctions multifractales ayant un spectre de singularités prescrit*. Comptes Rendus Acad. Sc. Paris, Série I, **315** (1992), 19–24.
- [20] S. Jaffard, *Formalisme multifractal pour les fonctions*. Comptes Rendus Acad. Sc. Paris, Série I, **317** (1993), 745–750.
- [21] S. Jaffard, *Multifractal Formalism for functions, Part I: Results valid for all functions, and Part II: Selfsimilar functions*. SIAM J. Math. Anal. **28** (1997) no. 4, 944–970 and 971–998.
- [22] S. Jaffard, *The spectrum of singularities of Riemann’s function*. Rev. Mat. Iberoam. **12**, No.2 (1996), 441–460.
- [23] Lévy Véhel J. and Vojak R., *Mutual multifractal analysis of sequences of Choquet capacities: preliminary results*. Adv. Applied Math. **20** (1998), 1–43.
- [24] B.B. Mandelbrot, *Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence*. In Statistical Models and Turbulence. Edited by M. Rosenblatt and C. Van Atta. Lectures Notes in Physics 12, Springer 1972, 331–351.
- [25] B.B. Mandelbrot, *Multiplications aléatoires itérées, et distributions invariantes par moyenne pondérée*. Comptes Rendus Acad. Sc. Paris, **278** (1974), 289–292 et 355–358.
- [26] B.B. Mandelbrot, *Fractal measures (their infinite moment sequences and dimensions) and multiplicative chaos*. Technical report, Physics Department, IBM Research Center, Mathematics Department, Harvard University, 1986.
- [27] B.B. Mandelbrot, *An introduction to multifractal distribution functions*. In Fluctuations and Pattern Formation (Cargese 1988). Edited by H. E. Stanley and N. Ostrowsky. Kluwer 1988, 345–360.
- [28] B.B. Mandelbrot, *Multifractal measures, especially for the geophysicists*. Pure and Applied Geophysics **131** (1989), 5–42.
- [29] B.B. Mandelbrot, *A class of multifractal measures with negative (latent) value for the dimension $f(\alpha)$* . In: Fractals: physical origin and properties (Erice, 1988). Edited by Luciano Pietronero, Plenum 1989, 3–29.
- [30] B.B. Mandelbrot, *Negative ELNA dimensions and ELNA Hölders, multifractals and their Hölder spectra, and the role of lateral pre-asymptotics in science*. In: J.-P. Kahane Meeting (Orsay, 1993). Journal of Fourier Analysis and Applications 1995.
- [31] R.H. Riedi and B.B. Mandelbrot, *Inversion formula for continuous multifractals* Adv. Appl. Math. **19** (1997), 332–354.
- [32] B.B. Mandelbrot and R.H. Riedi, *Inverse measures, the inversion formula and discontinuous Multifractals*. Adv. Appl. Math. **18** (1997), 50–58.
- [33] G. Michon, *Une construction des mesures de Gibbs sur certains ensembles de Cantor*. Comptes Rendus Acad. Sc. Paris, Série I, **308** (1989), 315–318.
- [34] G. Michon, *Mesures de Gibbs sur les Cantor réguliers*. Ann. IHP, Phys. Th. 58 n° 3 (1993), 267–285.
- [35] G. Michon et J. Peyrière, *Thermodynamique des ensembles de Cantor autosimilaires*. Chinese Ann. of Math. 15B (1994), 253–272.
- [36] L. Olsen, *A multifractal formalism*. Advances in Math. **116** (1995), 82–196.
- [37] L. Olsen, *Random geometrically graph directed self-similar multifractals*. Pitman Res. Notes Math. Ser., **307** (1994).
- [38] Y.B. Pesin, *Dimension Theory in Dynamical Systems*. Chicago Lectures in Mathematics. The University of Chicago Press, 1997.

- [39] J. Peyrière, *Multifractal measures*. In: Probabilistic and Stochastic Methods in Analysis (Proceedings of the NATO ASI, Il Ciocco 1991). Ed. J. Byrnes, Kluwer Academic Publishers (1992).
- [40] R. Riedi, Thesis, ETH Zurich 1993, Diss. ETH No 10077.
- [41] X. Saint Raymond and C. Tricot, *Packing regularity of sets in n -space*. Math. Proc. Cambridge Philos. Soc. **103** (1988), 133–145.
- [42] C. Tricot, *Two definitions of fractional dimension*. Math. Proc. Cambridge Philos. Soc. **91** (1982), 57–74.

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