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MULTIFRACTAL MEASURES

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ABSTRACT. The present redaction is mainly an account of a joint work [6] with G. Brown, from the University of New South Wales, and G. Michon, from Dijon University. The multifractal formalism is described, and a setting in which it holds is given, as well as the Michon construction of Gibbs measures.

INTRODUCTION: THE MULTIFRACTAL FORMALISM

Let ν_n be an increasing sequence of positive integers. The interval $[\frac{j}{\nu_n}, \frac{j+1}{\nu_n}[$ is denoted by $I_{n,j}$. Let μ be a probability measure on $[0,1[$. Set

$$\tau_n(q) = -\frac{1}{\log \nu_n} \log \sum_{0\le j<\nu_n}^{*} \mu(I_{n,j})^q$$

where $\sum^*$ means that the summation runs over the indices j such that $\mu(I_{n,j}) \neq 0$, and suppose that $\tau(q) = \lim_{n\to\infty} \tau_n(q)$ exists for every q in a certain interval $\mathcal{J}$ of $\mathbb{R}$.

On the other hand, let us define $I_n(x)$ to be the interval of the family $\{I_{n,j}\}_{0\le j<\nu_n}$ which contains the point x of $[0,1[$, and set, for $\alpha > 0$,

$$E_\alpha = \left\{ x \in [0,1[\,\Big|\, \lim_{n\to\infty} -\frac{\log \mu(I_n(x))}{\log \nu_n} = \alpha \right\}.$$

Then the multifractal formalism, as asserted in various works [11, 12, 13], and proved [1, 2, 5, 9, 15] to hold in various contexts, says that the Hausdorff dimension of E_α can be computed in the following way:

$$(*) \qquad \dim E_\alpha = \inf_{q\in\mathcal{J}} [\alpha q - \tau(q)].$$

In the case where τ is differentiable at a point q_0 and $\alpha = \tau'(q_0)$, we have $\dim E_\alpha = \alpha q_0 - \tau(q_0)$.

This article is organized as follows. In the next section, the setting is enlarged in order to deal with families of partitions the elements of which can have different lengths. Without any assumption on the measure, it is shown that the right hand side of $(*)$ is always an upper bound for $\dim E_\alpha$. Moreover, the result is a bit stronger, in the sense that we can deal with the Tricot (packing) dimension instead of Hausdorff's. We can also majorize the dimension of a larger set than E_α.

The third section is devoted to getting lower bounds for dimensions once the existence of Gibbs measures is assumed.

In the fourth section, Michon's proof of the existence of Gibbs measures for homogeneous trees is given.

Upper bounds for dimensions

Let $\{\{I_{n,j}\}_{1\le j\le\nu_n}\}_{n>0}$ be a sequence of partitions of $[0,1[$ by intervals, semi-open to the right. These partitions need not be nested. If $x \in [0,1[$, $I_n(x)$ stands for the interval of the family $\{I_{n,j}\}_{1\le j\le\nu_n}$ which contains x. The length of an interval J is denoted by $|J|$. We suppose that, for any $x \in [0,1[$, $\lim_{n\to\infty}|I_n(x)| = 0$.

We consider two indices dim and Dim which are defined as Hausdorff and Tricot dimensions are, but by only considering coverings and packings by intervals in the family $\{\{I_{n,j}\}_{1\le j\le\nu_n}\}_{n>0}$. An account of several notions of dimension is given in the appendix.

We are given a probability measure μ on $[0,1[$ and a sequence $\{\lambda_n\}_{n>0}$ of positive integers such that $\sum_{n>0}\exp(-\eta\lambda_n) < \infty$ for any $\eta > 0$.

We define the following quantities:

$$C_n(x,y) = \frac{1}{\lambda_n}\log \sum_{1\le j\le\nu_n}^{*} \mu_n(I_{n,j})^{x+1}\,|I_{n.j}|^{-y}$$

and

$$C(x,y) = \limsup_{n\to\infty} C_n(x,y)$$

where $\sum^*$ means that the summation runs over the j's such that $\mu_n(I_{n,j}) \neq 0$.

We suppose that $C(x,y)$ is not constantly equal to 0 or ∞ (this imposes the growth of the sequence $\{\lambda_n\}$), and set $\Omega = \{(x,y)\in\mathbb{R}^2 \,|\, C(x,y) < 0\}$. Since C is a convex function, non-decreasing as a function of x, and non-increasing as a function of y, there exists a concave and non-decreasing function φ from $\mathbb{R}$ to $\bar{\mathbb{R}}$ such that the interior of Ω is identical to the set $\{(x,y)\in\mathbb{R}^2 \,|\, y < \varphi(x-0)\}$. Of course, taking the limit to the left only matters at the left end of the interval $\mathcal{J}$ on which φ is finite. Besides, we assume that $0 \in \mathcal{J}$ and, for the sake of simplicity, that φ is differentiable on this interval (the complete discussion, in the case where it is not so, is given in [6]).

In the case described in the introduction, where all the intervals of the partition $\{I_{n,j}\}_{1\le j\le \nu_n}$ have the same length, $\lambda_n = \nu_n$ and the limit exists, we have $\varphi(x) = \tau(x+1)$, where τ is the function defined in the introduction.

Set $f(\alpha) = \inf\,[\alpha(x+1) - \varphi(x)]$.

On the other hand, we consider the following sets

$$\begin{aligned} B_\alpha &= \left\{x \in [0,1[\,\middle|\, \limsup \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \le \alpha\right\} \\ B^*_\alpha &= \left\{x \in [0,1[\,\middle|\, \liminf \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \le \alpha\right\} \\ V_\alpha &= \left\{x \in [0,1[\,\middle|\, \liminf \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \ge \alpha\right\} \\ V^*_\alpha &= \left\{x \in [0,1[\,\middle|\, \limsup \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \ge \alpha\right\} \end{aligned}$$

We then have the following result.

Theorem.

1. *For any α, we have* $\operatorname{Dim} B^*_\alpha \le -\varphi(-1)$ *and* $\operatorname{Dim} V^*_\alpha \le -\varphi(-1)$.
2. *If $\alpha \le \varphi'(-1)$, then* $\operatorname{Dim} B_\alpha \le f(\alpha)$ *and* $\dim B^*_\alpha \le f(\alpha)$.
3. *If $\alpha \ge \varphi'(-1)$, then* $\operatorname{Dim} V_\alpha \le f(\alpha)$ *and* $\dim V^*_\alpha \le f(\alpha)$.

Proof. Let us for instance consider the second case ($\alpha < \varphi'(-1)$), and set

$$B_\beta(n) = \left\{t \in [0,1[\,\middle|\, \mu(I_n(t)) \ge |I_n(t)|^\beta\right\}$$

We then have

$$B_\alpha = \bigcap_{\alpha<\beta<\varphi'(-1)} \bigcup_m \bigcap_{n\ge m} B_\beta(n).$$

Fix $\alpha < \beta < \varphi'(-1)$ and $\delta > f(\beta)$, and choose $t > 0$ such that $C(-1+t, -\delta + \beta t) < 0$. Then

$$\begin{aligned} \sum_{j:\mu(I_{n,j})\ge |I_{n,j}|^\beta} |I_{n,j}|^\delta &= \sum_{\text{idem}} |I_{n,j}|^{\beta t} |I_{n,j}|^{-(-\delta+\beta t)} \\ &\le \sum_j \mu(I_{n,j})^t |I_{n,j}|^{-(-\delta+\beta t)} \\ &\le \exp \lambda_n C_n(-1+t, -\delta+\beta t). \end{aligned}$$

Therefore

$$\sum_n \sum_{j:\mu(I_{n,j})\ge |In,j|^\beta} |I_{n,j}|^\delta < \infty.$$

So, if $\{I_j\}$ is a packing of $\bigcap_{n\geq m} B_\beta(n)$ by intervals from generations larger than m, we have $\sum |I_j|^\delta < \infty$. Therefore $\Delta\left(\bigcap_{n\geq m} B_\beta(n)\right) \leq \delta$ (cf. appendix) and $\operatorname{Dim} B_\alpha \leq \delta$. Finally $\operatorname{Dim} B_\alpha \leq f(\alpha)$.

The other cases are handled in a similar way.

An alternate definition of φ.

Consider the following quantity:

$$K(x,y) = \lim_{\varepsilon\to 0} \sup_{\varepsilon\text{-packing}} \sum \mu(I_j)^{x+1} |I_j|^{-y}.$$

The function K is convex, so the set $\Omega = \{K = 0\}$ is also convex. Moreover, if $K(a,b)$ is finite, then $K(a+t, y-u) = 0$ for positive t and u. Therefore, ther exists a concave and non-decreasing function φ from $\mathbb{R}$ to $\bar{\mathbb{R}}$ such that $\overset{\circ}{\Omega} = \{(x,y) \big| y < \varphi(x-0)\}$.

As previously, set $f(\alpha) = \inf_x \left(\alpha(x+1) - \varphi(x)\right)$. In these conditions, we have the following result.

Theorem.

1. *If $\alpha < \varphi'(-1)$, then* $\operatorname{Dim} B_\alpha \leq f(\alpha)$.
2. *If $\alpha > \varphi'(-1)$, then* $\operatorname{Dim} V_\alpha \leq f(\alpha)$.

Proof. In the first case ($\alpha < \varphi'(-1)$), if $\delta > f(\alpha)$ the straight half line of slope α stemming from the point $(-1, -\delta)$ intersects $\overset{\circ}{\Omega}$. In other terms, there exists a positive number t such that $(-1+t, -\delta+\alpha t) \in \Omega$. There exists $\varepsilon > 0$ such that, for any ϵ-packing $\{I_j\}_{j>0}$ of $[0,1[$ by elements of the family $\{I_{n,j}\}_{n,j}$ we have $\sum_j \mu(I_j)^{x+1} |I_j|^{-y} \leq 1$.

As in the preceding section, we write

$$B_\alpha^* = \bigcap_{\alpha<\beta<\varphi'(-1)} \bigcap_{m\geq 1} \bigcup_{n\geq m} B_\beta(n).$$

Therefore, if n is such that $\sup_j |I_{n,j}| \leq \varepsilon$, if $\alpha < \beta < \varphi'(-1)$, and if $\{I_j\}_j$ is an ε-packing of the set $B_\beta(n)$, we have

$$\begin{aligned}\sum |I_j|^\delta &= \sum |I_j|^{t\alpha} |I_j|^{\delta - t\alpha}\\ &\leq \sum \mu(I_j)^{(-1+t)+1} |I_j|^{-(-\delta+\alpha t)} \leq 1\end{aligned}$$

So, $\operatorname{Dim} B_\beta(n) \leq \delta$, and $\operatorname{Dim} B_\alpha \leq f(\alpha)$.

The second case is handled in the same way.

Remark. We could also have defined $K(x,y)$ to be $\lim_{\varepsilon\to 0}\inf\sum\mu(I_j)^{x+1}|I_j|^{-y}$ where the inf is taken over the ε-coverings $\{I_j\}$ of $[0,1[$ by elements of the family $\{I_{n,j}\}_{n,j}$. The function K may be no longer convex, but the boundary of $\{K=0\}$ is still defined by a non-decreasing function φ from $\mathbb{R}$ to $\bar{\mathbb{R}}$. If f is defined as above, then a similar conclusion hold by replacing Dim by dim.

Lower bounds for dimensions

The notations are the same as in the previous section. If u and v are two functions, the relation $u\approx v$ means that there exists a positive constant K such that $K^{-1}u\le v\le Ku$.

Theorem. *Let $\theta\in\mathbb{R}$ and suppose that $\varphi'(\theta)$ exists and that there is a measure μ_θ such that $\mu_\theta(I_{n,j})\approx\mu(I_{n,j})^{\theta+1}|I_{n,j}|^{-\varphi(\theta)}$. Then we have $\dim E_{\varphi'(\theta)}=f(\varphi'(\theta))$.*

The measure μ_θ, in analogy with the statistical mechanics, is called a Gibbs measure.

Proof. Consider the following quantities

$$\widetilde{C}_n(x,y)=\frac{1}{\lambda_n}\log\sum_j^{*}\mu(I_{n,j})^x|I_{n,j}|^{-y}\mu_\theta(I_{n,j})$$

and

$$\widetilde{C}(x,y)=\limsup\widetilde{C}_n(x,y).$$

We have

$$\widetilde{C}_n(x,y)=\frac{1}{\lambda_n}\log\sum^{*}\mu(I_{n,j})^{x+\theta+1}|I_{n,j}|^{-y-\varphi(\theta)}+o(1)$$

and

$$\widetilde{C}(x,y)<0\Leftrightarrow C(x+\theta,y+\varphi(\theta))<0.$$

Therefore

$$\left\{\widetilde{C}<0\right\}^{\circ}=\left\{(x,y)\big|y<\varphi(x+\theta)-\varphi(\theta)\right\}.$$

Lemma. *As n goes to infinity, $\frac{\log \mu(I_n(t))}{\log |I_n(t)|} \to \varphi'(\theta)$ for μ_θ-almost every t.*

Proof. If $\alpha < \varphi'(\theta)$ then there exists $t > 0$ such that $\widetilde{C}(t, \alpha t) < 0$. Then

$$\begin{aligned} \mu_\theta \left\{ t \middle| \mu(I_n(t)) > |I_n(t)|^\alpha \right\} &= \sum_{j : \mu(I_{n,j}) > |I_{n,j}|^\alpha} \mu_\theta(I_{n,j}) \\ &= \sum_{\text{idem}} |I_{n,j}|^{\alpha t} |I_{n,j}|^{-\alpha t} \mu_\theta(I_{n,j}) \\ &\leq \sum_j \mu(I_{n,j})^t |I_{n,j}|^{-\alpha t} \mu_\theta(I_{n,j}) = \exp \lambda_n \widetilde{C}_n(t, \alpha t). \end{aligned}$$

(this inequality is a large deviation type result [7], as well as the analogous one in the proof of the theorem concerning upper bounds).
Therefore

$$\sum_n \mu_\theta \left\{ \frac{\log \mu(I_n(t))}{\log |I_n(t)|} < \alpha \right\} < \infty$$

so, $\liminf \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \geq \alpha$ for μ_θ-almost t. The upper limit is treated similarly.

We can now end the proof of the theorem. It result from the above lemma first that $\mu_\theta\big(E_{\varphi'(\theta)}\big) = 1$ and secondly, taking into account the properties of μ_θ, that

$$\frac{\log \mu_\theta(I_n(t))}{\log |I_n(t)|} \to (\theta + 1)\varphi'(\theta) - \varphi(\theta)$$

for μ_θ-almost t. Therefore, due to the Billingsley-Kinney-Pitcher theorem, we have $\dim E_{\varphi'(\theta)} \geq f(\varphi'(\theta))$. The equality then results from the previous section.

As a consequence, the Hausdorff or Tricot dimensions of all the sets E_α, B_α, V_α, B^*_α, and V^*_α are equal to $f(\alpha)$ under the same conditions. This generalizes some results of Besicovitch [3], Eggleston [10], and Volkman [28] on the dimension of sets defined in terms of frequency of digits. This also accounts for some results in [9] and some studies on 'cookie-cutters' [2, 5].

Existence of Gibbs measures

In this section we suppose that the sequence $\{\{I_{n,j}\}_{1 \leq j \leq \nu_n}\}_{n>0}$ of partitions has the following properties: each element of the (n+1)-th partition is contained in one element of the n-th one, and each element of the n-th partition is split into a fixed number p of elements of the (n+1)-th one. Obviously, this imposes $\nu_n = p^n$. We are going to use another indexation of the intervals $\{I_{n,j}\}$: the intervals $\{I_{2,j}\}$ will be denoted by $I_{i_1 i_2}$, with $0 \leq i_1, i_2 < p$, in such a way that $I_{i_1 i_2} \subset I_{i_1}$; and so on.

Let $\mathcal{A}$ be the set of words over the alphabet $\{0, 1, \ldots, p-1\}$. The concatenation, just denoted by juxtaposition, endows $\mathcal{A}$ with a structure of semigroup. The empty word, which is the unit, is denoted by ϵ. The set of words ot length n is denoted by $\mathcal{A}_n$; it indexes the elements of the n-th partition. If $a \in \mathcal{A}$, instead of writing $\mu(I_a)$ we shall simply write $\mu(a)$. In these conditions, for every $a \in \mathcal{A}$, we have $\sum_{0\leq b<p} \mu(ab) = \mu(a)$.

We suppose that μ is quasi-Bernoulli, i.e. there exists a positive number M such that, for any a and b in $\mathcal{A}$, we have $M^{-1}\mu(a)\mu(b) \leq \mu(ab) \leq M\mu(a)\mu(b)$.

We also define a mapping l from $\mathcal{A}$ to $\mathbb{R}$: $l(a) = |I_a|$. We assume that l is almost multiplicative, i.e. there exists a positive constant L such that, for any a and b in $\mathcal{A}$, we have $L^{-1}l(a)l(b) \leq l(ab) \leq L\, l(a)l(b)$.

Under these conditions, G. Michon [21, 22] proved that the 'free energy' exists and that there are Gibbs measures. We are going now to give his proof.

Proposition. *For every x and y in $\mathbb{R}$, the ratio $\frac{1}{n}\log[\sum_{a\in\mathcal{A}_n} l(a)^{-y}\mu(a)^{x+1}]$ has a limit, denoted by $C(x,y)$, as n goes to infinity.*

Proof. By replacing l by $l^{-y}\mu^x$, it is enough to consider the case $x = 0$, $y = -1$. Set

$$Z_n = \sum_{a\in A_n} l(a)\mu(a), \text{ and } \ C_n = \frac{1}{n}\log(Z_n).$$

We have

$$Z_{m+n} = \sum_{a\in\mathcal{A}_m} l(a)\mu(a) \sum_{b\in\mathcal{A}_n} \frac{l(ab)}{l(a)l(b)}\frac{\mu(ab)}{\mu(a)\mu(b)} l(b)\mu(b).$$

Therefore, we have $|\log Z_{m+n} - \log Z_m - \log Z_n| \leq \log(ML)$. It results that C_n has a limit C as n goes to infinity. Moreover, we have $|C_n - C| \leq \frac{1}{n}\log ML$.

Let us notice that if we set $l_a(b) = l(ab)/l(a)$, (for a and b in $\mathcal{A}$), we have $L^{-2}l_a(b)l_a(c) \leq l_a(bc) \leq L^2 l_a(b)l_a(c)$ for a, b, and c in $\mathcal{A}$. Similarly for μ.

For any a in $\mathcal{A}$, and s in $\mathbb{R}$, set ${}_aZ_n = \sum_{b\in\mathcal{A}_n} l_a(b)\,\mu_a(b)$, and

$$(**) \qquad Z_a(s) = \sum_{n\geq 0} {}_aZ_n e^{-ns}$$

It results from the above remark that, for any n and for any a, we have $K^{-1}\,{}_\epsilon Z_n \leq {}_aZ_n \leq K\,{}_\epsilon Z_n$, with $K = L\,M$. Therefore $\lim_{n\to\infty}\frac{1}{n}\log{}_aZ_n$ does not depend on a and is equal to what we called C in the proof of the above proposition. Moreover, $\left|\frac{1}{n}\log{}_aZ_n - C\right| \leq \frac{2}{n}\log K$ and $K^{-2}\exp nC \leq {}_aZ_n \leq K^2\exp nC$ for any n. So the series $(**)$ converges for $s > C$. From these last inequalities it results also that

$$\frac{K^{-2}}{1-\exp(C-s)} \leq Z_a(s) \leq \frac{K^2}{1-\exp(C-s)}$$

Theorem. *For every x and y in $\mathbb{R}$, there exist a constant c and a measure $\mu_{x,y}$ such that, for any a in $\mathcal{A}_n$, we have*

$$c^{-1}l(a)^{-y}\,\mu(a)^{x+1}\,e^{-nC(x,y)} \leq \mu_{x,y}(a) \leq c\,l(a)^{-y}\,\mu(a)^{x+1}\,e^{-nC(x,y)}$$

Proof. As previously, it is enough to consider the case $x = 0,\ y = -1$.

Let us denote by l_n the following mapping from $[0,1[$ to $\mathbb{R}$: $l_n(t) = |I_n(t)|$. Let us define a family of functions from $[0,1]$ to $\mathbb{R}^+$ in the following way $\varphi_s = l_0 + l_1 e^{-s} + l_2 e^{-2s} + \cdots$. Obviously, we have $\int \varphi_s \, d\mu = Z_\epsilon(s)$. This allows us to define the family $P_s = \frac{\varphi_s}{Z_\epsilon(s)}\mu \quad (s > C)$ of probability measures on $[0,1]$.

If $a \in \mathcal{A}_n$, we denote by $a_j \quad (0 \le j \le n)$ the word formed by the j first letters of a. In these conditions, we have, denoting $P_s(I_a)$ simply by $P_s(a)$,

$$Z_\epsilon(s)\, P_s(a) = \mu(a) \sum_{0 \le j < n} l(a_j)\, e^{-js} + \sum_{j \ge 0} \sum_{b \in \mathcal{A}_j} l(ab)\, \mu(ab)\, e^{-(j+n)s}$$

In other terms

$$P_s(a) = \frac{\mu(a)}{Z_\epsilon(s)} \sum_{0 \le j < n} l(a_j)\, e^{-js} + l(a)\, \mu(a)\, e^{-ns} \frac{Z_a(s)}{Z_\epsilon(s)}.$$

When s goes to C, P_s has a weak limit point γ at least. But, we know that, as s goes to C, $Z_\epsilon(s)$ goes to infinity and that the ratio $Z_a(s)/Z_\epsilon(s)$ stays between K^{-1} and K. This means that we have, for $a \in \mathcal{A}_n$,

$$K^{-2} \le \frac{\gamma(a)}{l(a)\, \mu(a)\, e^{-nC}} \le K^2$$

Remark. The case of Riesz products [8] is not handled by this proof.

Example

One of the paradigms of mutifractality is the multinomial measures of which we give a generalization in this section..

Let X be the simplex $\{(x_1, \ldots, x_p) \big| x_1 + \cdots + x_p = 1,\ x_j \ge 0 \text{ for } j = 1, \ldots, p\}$, $(p \ge 2)$. Consider a sequence $\{m_n, l_n\}_{n>0}$ of elements of $X \times X$. We assume that this sequence has a continuous measure of repartition ξ. This means that there exists a continuous probability measure ξ on the space $X \times X$ such that for any open set $U \subset X \times X$ the boundary of which is of zero ξ-measure we have

$$\lim_{n \to \infty} \frac{1}{n} \# \{j \le n \big| (m_j, l_j) \in U\} = \xi(U).$$

Moreover we assume that the boundary of $X \times X$ is of zero ξ-measure.

As in the section concerning the construction of Gibbs measures, we consider subintervals of $[0,1[$ indexed by words over the alphabet $\{0, 1, \ldots, p-1\}$: $I_\epsilon = [0,1[$, and the length of $I_{x_1 \ldots x_{n+1}}$ is $l_{n+1, x_{n+1}} \left| I_{x_1 \ldots x_n} \right|$.

Define a measure μ on $[0,1[$ in the following way:

$$\mu(I_{x_1,\ldots,x_n}) = \prod_{1\le j\le n} m_{j,x_j}.$$

It can be verified that we have

$$\lim_{n\to\infty} \frac{1}{n} \log \sum \mu(I_{x_1,\ldots,x_n})^{x+1} |I_{x_1,\ldots,x_n}|^{-y} = \int_{X\times X} \log \sum_{1\le k\le p} \left(u_k^{x+1} v_k^{-y}\right) d\xi(u,v).$$

So we have an explicit expression for $C(x,y)$. On the other hand, if we keep the same l_n, but if m_n is replaced by $\frac{m_n^{x+1} l_n^{-y}}{\sum_{1\le k\le p} m_{n,k}^{x+1} l_{n,k}^{-y}}$ (where to exponentiate a vector means to exponentiate each of its components) and perform a similar construction, we get a measure $\nu_{x,y}$ which is the Gibbs measure corresponding to (x,y). Therefore in this situation the multifractal formalism holds.

Appendix: Hausdorff and Tricot dimensions

Hausdorff dimension.

Let E be a subset of $[0,1[$ and α a positive number. Set

$$H_\alpha(E) = \lim_{\varepsilon\to 0} \inf \left\{ \sum |I_j|^\alpha \,\middle|\, E \subset \bigcup I_j,\ |I_j| < \varepsilon \right\}.$$

If $H_\alpha(E) < \infty$ then $\beta > \alpha \Rightarrow H_\beta(E) = 0$. So there is a cutoff α_0 such that

$$\alpha < \alpha_0 \Rightarrow H_\alpha(E) = \infty \quad \text{and} \quad \alpha > \alpha_0 \Rightarrow H_\alpha(E) = 0.$$

This number α_0 is, by definition, the Hausdorff dimension of E.

An other dimensional index is of wide use. Let $N_\varepsilon(E)$ be the minimum number of elements of coverings of E by intervals of lengths less than ε, and set

$$\Delta(E) = \limsup_{\varepsilon\to 0} -\frac{\log N_\varepsilon(E)}{\log \varepsilon}.$$

This index has been considered by many authors and bears several names: Bouligand-Minkowski dimension, entropy dimension, logarithmic index, box dimension ... In fact these indices differ in a general metric space. Obviously we have $\dim E \le \Delta(E)$.

The following observation gives a way of getting a lower bound for the Hausdorff dimension: if there exists a measure μ satisfying a Hölder condition of order α (i.e. $\mu(I) \le C|I|^\alpha$ for every interval I) and such that $\mu(E) > 0$, then $\dim E \ge \alpha$. Indeed if $\{I_j\}$ is a covering of E by intervals, we have

$$0 < \mu(E) \le \sum \mu(I_j) \le C \sum |I_j|^\alpha$$

wich proves the above assertion. In fact a refinement of this argument gives the following lemma due to Kinney and Pitcher and, in a more general form, to Billingsley [4].

Lemma. *Let μ be a probability measure. If*

$$\mu(E) > 0 \text{ and } E \subset \left\{ t \big| \liminf_{I \searrow \{t\}} \frac{\log \mu(I)}{\log |I|} \geq \alpha \right\}$$

then $\dim E \geq \alpha$.

Tricot dimension.

An ε-packing of E is a collection of mutually disjoint intervals intersecting E. The following property of box dimension can be found in [26]:

$$\begin{aligned} \Delta(E) &= \inf \left\{ \alpha \big| \lim_{\varepsilon \searrow 0} \sup \left\{ \sum |I_j|^\alpha \big| \{I_j\} \text{ being an } \varepsilon\text{-packing of } E \right\} = 0 \right\} \\ &= \sup \left\{ \alpha \big| \lim_{\varepsilon \searrow 0} \sup \left\{ \sum |I_j|^\alpha \big| \{I_j\} \text{ being an } \varepsilon\text{-packing of } E \right\} = \infty \right\} \end{aligned}$$

One drawback of the box dimension is that it does not distinguish a set from its closure. For instance, it assigns 0 to the dimension of the rationals. This led Tricot [25, 26] to introduce the following concept:

$$\operatorname{Dim} E = \inf \left\{ \sup_n \Delta(E_n) \big| E \subset \bigcup E_n \right\}$$

Obviously $\dim E \leq \operatorname{Dim} E$. A related notion has been introduced by Sullivan [24]. An account of this notion of dimension and of connected outer measures can be found in [27].

The index Dim has the same regularity properties as Hausdorff dimension: if $E \subset F$ then $\operatorname{Dim} E \leq \operatorname{Dim} F$, and, if E is the union of a non-decreasing sequence $\{E_n\}$ of sets, then $\operatorname{Dim} E = \sup_n \operatorname{Dim} E_n$.

Final remarks

There are many developments, which we are not going to discuss, about multifractals, in particular concerning further interpretations [16–20] of the function $f(\alpha)$ especially when it assumes negative values.

In a recent work Muzy et al. [23] have adapted this formalism to handle another situation by replacing indicator functions of intervals by wavelets.

Besides, the thermodynamical formalism has been used to study harmonic measures [14].

Bibliography

[1] BEDFORD, T. *Hausdorff dimension and box dimension in self-similar sets.* Proc. conf. Topology and measure V. Binz, GDR, 1987).

[2] BEDFORD, T. *Applications of dynamical systems theory to fractals. A study of cookie-cutter Cantor sets.* Preprint TU Delft (Netherlands).

[3] BESICOVITCH, A. S. *On the sum of digits of real numbers represented in the dyadic system.* Math. Ann. 110 (1934), 321-330.

[4] BILLINGSLEY, P. *Ergodic theory and information.* J. Wiley 1965.

[5] BOHR, T. & RAND, D. *The entropy function for characteristic exponents.* Physica 25D (1987), 387-398.

[6] BROWN, G., MICHON, G. & PEYRIERE, J. *On the multifractal analysis of measures.* Submitted to J. Stat. Phys.

[7] CHERNOFF, H. *A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations.* Ann. Math. Statist. 23 (1952), 493-507.

[8] COLLET , P. Preprint.

[9] COLLET, P., LEBOWITZ, J. L. & PORZIO, A. *The dimension spectrum of some dynamical systems.* J. Stat. Phys. 47 (1987), 609-644.

[10] EGGLESTON, H. G. *The fractional dimension of a set defined by decimal properties.* Quart. J. Math. Oxford, ser. 20 (1949), 31-46.

[11] FRISCH, U. & PARISI, G. *Fully developped turbulence and intermittency in turbulence, and predictability in geophysical fluid dynamics and climate dymnamics.* International school of physics "Enrico Fermi", course 88, edited by M. Ghil, North Holland (1985), p. 84.

[12] HALSEY, T. C., JENSEN, M. H., KADANOFF, L. P., PROCACCIA, I. and SHRAIMAN, B. I. *Fractal measures and their singularities : the characterisation of strange sets.* Phys. Rev. A 33 (1986), p. 1141.

[13] HENTSCHEL, H. G. E. & PROCACCIA, I. *The infinite number of generalized dimensions of fractals and strange attractors.* Physica 8D (1983), p. 435.

[14] MAKAROV, N. Preprint

[15] MANDELBROT, B. B. *Multifractal measures, especially for the geophysicist.* Annual reviews of materials sciences 19., 1989, 514-516.

[16] MANDELBROT, B. B. *A class of multifractal measures with negative (latent) value for the dimension f(alpha).* Fractals : physical origin and properties (Erice, 1988). Edited by Luciano Pietronero, New York : Plenum, 1989.

[17] MANDELBROT, B. B. *Two meanings of multifractality, and the notion of negative fractal dimension.* Soviet-American chaos meeting (Woods Hole, 1989). Edited by Kenneth Ford and David Campbell, American Institute of Physics, 1990.

[18] MANDELBROT, B. B. *Limit lognormal multifractal measures.*Frontiers of physics : Landau memorial conference (Tel Aviv, 1988). Edited by Errol Gotsman, New York : Pergamon, 1989, 91-122.

[19] MANDELBROT, B. B. *New "anomalous" multiplicative multifractals : left sided f(alpha) and the modeling of DLA*. Condensed matter physics, in honour of Cyrill Domb (Bar Ilan, 1990). Physica A, 1990.

[20] MANDELBROT, B. B., EVERTSZ, C. J. G., & HAYAKAWA, Y. *Exactly self-similar "left-sided" multifractal measures, (with Carl J. G. Evertsz and Y. Hayakawa)*. Submitted to Physical Review A.

[21] MICHON, G. *Une construction des mesures de Gibbs sur certains ensembles de Cantor.* C. R. Acad. Sc. Paris 308 (1989), 315-318.

[22] MICHON, G *Mesures de Gibbs sur les Cantor Réguliers.* Submitted to Ann. IHP série B.

[23] MUZY, J.-F., BACRY, E., & ARNEODO, A. *Wavelets and multifractal formalism for singular signals: application to turbulence data.* Preprint.

[24] SULLIVAN, D. *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.* Acta Math. 153 (1984), 259-277.

[25] TRICOT, C. *Sur la classification des ensembles boréliens de mesure de Lebesgue nulle.* Thèse (n° 1921) présentée à la Faculté des Sciences de l'Université de Genève (1980).

[26] TRICOT, C. Jr *Two definitions of fractional dimension.* Math. Proc. Cambridge Philos. Soc. 91 (1982), 57-74.

[27] TRICOT, C. & TAYLOR, S. J. *Packing measure and its evaluation for a brownian path.* Trans. Amer. Math. Soc. 288 (1985) n° 2, 679-699.

[28] VOLKMAN, B. *Ober Hausdorffsche dimensionen von Mengen, die durch Zifferneigenschaften charakterisiert II, III & IV.* Math. Zeitschr. 59 (1953-54), 247-254, 259-270, 425-433.