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N17

"On certain martingales of Benoit Mandelbrot" Guest contribution (Kahane & Peyrière 1976)

✦ **Abstract.** Following his critical analysis of the random model of turbulence due to A. M. Yaglom, M 1974f{N14} and M1974c{N15} introduced his own model, which he calls "canonical." It proceeds from a brick, that is subsequently divided into $b, b^2, \ldots, b^n, \ldots$ similar bricks; each brick of the n-th stage is divided into b equal bricks in the $(n+1)$-th stage. Also given is a sequence of random variables W_p, which are independent, identically distributed, positive, have mean 1 and are indexed by the bricks P under consideration. Starting from the Lebesgue measure μ_0 on the initial brick, one constructs the sequence of measures μ_n by successive stages. Thus, μ_n has a constant density on each brick P of the n-th stage, and the density of μ_n on P is the product of W_P and the density μ_{n-1} on P. The sequence of measures μ_n is a vector martingale, and it converges towards a random measure μ. M 1974c gives results and raises problems concerning the measure μ: non-degeneracy, the moments of $||\mu||$, the Borel sets supporting μ and their Hausdorff dimension. Some of the conjectures of Mandelbrot have been solved by Kahane 1974 or by Peyrière 1974. Here we present these results in a refined form. Theorems 1, 2 and 3 below are due to J.-P. Kahane, Theorem 4 is due to J. Peyrière. ✦

1. Introduction, definitions, main results and history

It will be convenient to take as the initial brick the interval $[0, 1]$. The "bricks" P are then the b-adic intervals for $n = 1, 2, \ldots$ and $j_k = 0, \ldots, b-1$, namely

$$I(j_1, j_2, \ldots, j_n) = \left[\sum_1^n j_k b^{-k}, \sum_1^n j_k b^{-k} + b^{-n}\right[.$$

Given an integer $b \geq 2$, and a positive random variable (r.v.) W with $E(W) = 1$, one denotes by $W(j_1, j_2, \ldots, j_n)$ a sequence of independent r.v.'s, having the same distribution as W, and one denotes by μ_n the measure defined on $[0, 1]$, whose density on the interval $I(j_1, j_2, \ldots, j_n)$ is given by $W(j_1)W(j_1, j_2) \ldots W(j_1, j_2, \ldots, j_n)$. Let

$$Y_n = \|\mu_n\| = b^{-n} \sum_{j_1, j_2, \ldots, j_n} W(j_1)W(j_1, j_2) \ldots W(j_1, j_2, \ldots, j_n). \tag{1}$$

This is a nonnegative martingale, with $E(Y_n) = 1$. Hence, it converges almost surely (a.s.) towards a r.v. Y_∞ such that $E(Y_\infty) \leq 1$. In the same fashion, for all b-adic intervals I, $\mu_n(I)$ is a martingale with expectation $|I|$ which converges a.s. to a limit $\mu(I)$. Hence μ_n tends weakly a.s. to a measure μ of total mass Y_∞.

It is convenient to write (1) in the form

$$Y_n = b^{-1} \sum_{j=0}^{b-1} W(j) Y_{n-1}(j). \tag{2}$$

The r.v. $W(j)$ and $Y_{n-1}(j)$ are mutually independent, and the $Y_{n-1}(j)$ have the same distribution as Y_{n-1}.

Consider finally the functional equation

$$Z = b^{-1} \sum_{j=0}^{b-1} W_j Z_j, \tag{3}$$

where the r.v.'s W_j and Z_j are mutually independent, the W_j having the same distribution as W, and the Z_j having the same distribution as Z. The unknown in (3) is the distribution of Z; by an abuse of language, Z will be called solution of (3). Other solutions may exist; for example, in the case $W \equiv 1$, a Cauchy variable is a solution to (3), and it cannot be of type Y_∞ because it is neither positive nor integrable.

It will be convenient to associate with W the convex function

$$\varphi(q) = \log_b E(W^q) - (q - 1). \tag{4}$$

It is always defined for $0 \leq q \leq 1$, and can be defined for values $q > 1$. The function φ is zero at the point 1, and at most at one other point, q_{crit}. The left-side derivative of φ at the point 1 is

$$\varphi'(1-0) = E(W \log_b W) - 1 = -D \text{ (by definition of } D).$$

We will see the role that D plays in the non-degeneracy of μ, and in the dimension of the Borel sets supporting μ. Also, we shall see the role of q_{crit} with respect to the moments of Y_∞.

The most striking illustrations are the following. (1) $W = e^{\tau\xi - (\tau^2/2)}$, where ξ is a normal variable (this is the origin of theory); then φ is a polynomial of degree 2. (2) W has only two possible values, one of which is zero; then φ is a linear function, and $b^n Y_n$ may be interpreted as the population at time n in a birth-and-death process in which each individual gives birth to b descendants, whose probability of survival is $P(W \neq 0)$.

All these notions were introduced in M 1974f{N15} and M 1974c{N16}.

We will establish the following results.

Theorem 1. *The following statements provide equivalent conditions of non-degeneracy:*

(α) $E(Y_\infty) = 1$,
(β) $E(Y_\infty) > 0$,
(γ) equation (3) has a solution Z such that $E(Z) = 1$,
(δ) $E(W \log W) < \log b$.

Theorem 2. (Condition for the existence of finite moments). *Let* $q > 1$. One has $0 < E(Y_\infty^q) < \infty$ if and only if $E(W^q) < b^{q-1}$.

Theorem 3. (Case where Y_∞ has moments of every order). (1) *The following statements are equivalent:* (α_1) $0 < E(Y_\infty^q) < \infty$ *for all* $q > 1$; (β_1) $||W||_\infty =$ ess. sup adjust(u 2)$W \leq b$ *and* $P(W = b) < 1/b$ (strict inequality).

(2) If (β_1) holds, one has

$$\lim_{q\to\infty} \frac{\log E(Y_\infty^q)}{q \log q} = \log_b ||W||_\infty. \tag{5}$$

Theorem 4. (Study of the measure μ). *Suppose* $E(Y_\infty \log Y_\infty) < \infty$. *For each* $x \in [0, 1[$, *denote by* $I_n(x)$ *the b-*adic *interval of order n that contains x; its Lebesgue measure is* $m(I_n(x)) = b^{-n}$. *One has*

$$\mu\text{-almost everywhere}\ ,\ \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log m(I_n(x))} = D = 1 - E(W \log_b W) \qquad (6)$$

Corollary. *The measure* μ *is a.s. supported by a Borel set of Hausdorff dimension D, while all Borel sets of Hausdorff dimension* $< D$ *have* μ*-measure zero.*

Historical remarks. Condition (δ) of Theorem 1 can be written as $D > 0$. Kahane 1974 had only shown that

$$D > 0 \Rightarrow (\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow D \geq 0.$$

The role of D in the study of degeneracy had been guessed in M 1974c{N16}, Section 10.

Theorem 2 was conjectured in M 1974c and proved in Kahane 1974. The proof that we will give is simpler. Let us remark that the condition $E(W^q) < b^{q-1}$ can also be written as $\varphi(q) < 0$. If φ is zero for some $q_{crit} > 1$, this means $q < q_{crit}$.

Theorem 3 constitutes a critical comment on M 1974c, Proposition 10. It corresponds to $q_{crit} = \infty$. The proof will give some variants of Kahane 1974.

The Corollary of Theorem 4 confirms a conjecture of M 1974c{N16}, and improves on Peyrière 1974.

2. Proof of Theorem 1

Obviously (α) $\Rightarrow$ (β) $\Rightarrow$ (γ). Assume (γ), and let Z be a solution of (3) such that $E(Z) = 1$. There exists a sequence of independent r.v.'s $W(j_1, j_2, \dots, j_n)$ ($n = 1, 2, \dots ; j_k = 0, 1, \dots, b-1$), having the same distribution as W, and a sequence of r.v.'s $Z(j_1, j_2, \dots, j_n)$ with the same distribution as Z and independent of the $W(i_1, i_2, \dots, i_n)$ for $k \leq n$, such that for all n

$$Z = b^{-n} \sum_{j_1, \dots, j_n} W(j_1) W(j_1, j_2) \dots W(j_1, j_2, \dots, j_n) Z(j_1, j_2, \dots, j_n). \qquad (7)$$

Indeed, (7) reduces to (3) for $n=1$ ($W(j)=W_j$ and $Z(j)=Z_j$), and equation (3), if applied to $Z(j_1, j_2, \dots, j_n)$, gives

$$Z(j_1, j_2, \dots, j_n) = b^{-1}\sum_{j_{n+1}} W(j_1, j_2, \dots, j_{n+1})Z(j_1, j_2, \dots, j_n, j_{n+1}),$$

with the required conditions for the r.v.'s on the right hand side. The conditional expectation of Z with respect to the σ-field generated by the $W(j_1, \dots, j_k)$ $(k \le n)$ is Y_n as defined by (1). It follows that the martingale Y_n is uniformly integrable and that a.s. $Z = Y_\infty$ (see for example Meyer 1966, Section V8). Hence $(\gamma) \Rightarrow (\alpha)$, and furthermore (γ) implies $Z \ge 0$ a.s..

Assume again (γ), and consequently $Z \ge 0$. For $0 < q < 1$, the function x^q is sub-additive, hence (3) yields

$$E(b^q Z^q) \le \sum_{j=0}^{b-1} E[(W_j Z_j)^q] = bE(W^q)E(Z^q), \tag{8}$$

with $0 < E(Z^q) \le 1$. Therefore, the function $\varphi(q)$ as defined by (4) is nonnegative on [0, 1], from which it follows that $\varphi'(1-0) \le 0$, that is $D \ge 0$. To go further, (8) needs to be improved.

Lemma A. $(x+y)^q \le x^q + qy^q$ *for* $x \ge y > 0$, *and* $0 < q < 1$.

Proof. Use the fact that $y = 1$ and the formula of finite increments.

Lemma B. *Let X be a positive integrable r.v., and X' a r.v. with the same distribution as X and independent of X.* There exists a number $\varepsilon_X > 0$ such that

$$E(X^q 1_{X' \ge X}) \ge \varepsilon_X E(X^q) \quad \text{for } 0 \le q \le 1.$$

Proof. Each of these expectations is a continuous function of h, and is strictly positive on [0, 1].

Since the function x^q is sub-additive, it follows from (3)

$$b^q Z^q \le \sum_{j=0}^{b-1} W_j^q Z_j^q \quad \text{a.s.}$$

From Lemma A,

$$b^q Z^q \leq q W_0^q Z_0^q + \sum_{j=1}^{b-1} W_j^q Z_j^q \quad \text{if} \quad W_1 Z_1 \geq W_0 Z_0,$$

hence

$$E(b^q Z^q) = \sum_{j=0}^{b-1} E(W_j^q Z_j^q) - (1-q) E(W_0^q Z_0^q 1_{W_1 Z_1 \geq W_0 Z_0}),$$

which gives, using Lemma B,

$$E(b^q Z^q) \leq b E(W^q) E(Z^q) - (1-q) \varepsilon_{WZ} E(W^q) E(Z^q). \tag{9}$$

(9) is the desired refinement of (8). Dividing by $E(Z^q)$ and taking logarithms, and writing $\varepsilon = \varepsilon_{WZ}$, one has

$$\varphi(q) + \log_b(1 - \frac{(1-q)\varepsilon}{b}) \geq 0 \quad \text{on } [0, 1],$$

from which it follows that $\varphi'(1-0) + (\varepsilon / b \log b) \leq 0$, hence $D > 0$.

We have already shown that (α) $\Leftrightarrow$ (β) $\Leftrightarrow$ (γ) $\Rightarrow$ (δ). We will finish the demonstration by showing that (δ) implies (β)

Lemma C. $(x+y)^q \geq x^q + y^q - 2(1-q)(xy)^{q/2}$ *for* $x > 0$, $y > 0$, $q_0 < q < 1$.

Proof. One verifies that the function $f(t) = e^{tq} + e^{-tq} - (e^t + e^{-t})^q$ is strictly decreasing, hence has a maximum at $t = 0$, where $f(1) = 2 - 2^q$ and $f'(1) = -2\log 2 < 1$. This establishes the lemma.

Here is a corollary: one has

$$\left(\sum_1^b x_j \right)^q \geq \sum_1^b x_j^q - 2(1-q) \sum_{i<j} (x_i x_j)^{q/2} \tag{10}$$

for $x_j > 0$ ($j = 1, 2, \dots, b$) and $q_0 < q < 1$. Indeed, (10) is obtained by induction from

$$\left(\sum_1^b x_j\right)^q \geq x_1^q + \left(\sum_2^b x_j\right)^q - 2(1-q)x_1^{q/2}\left(\sum_2^b x_j\right)^{q/2}$$

$$\geq x_1^q + \left(\sum_2^b x_j\right)^q - 2(1-q)\sum_{j>1}(x_1 x_j)^{q/2}$$

which results from Lemma C and the sub-additivity of the function $x^{q/2}$.

Let us return to formula (2) and rewrite it provisionally as

$$Y = b^{-1}\sum_{j=0}^{b-1} W_j X_j, \tag{11}$$

where Y, W_j, X_j stand for Y_n, $W(j)$, $Y_{n-1}(j)$. Suppose that $q_0 < q < 1$. Apply Lemma C in the form (10) with $x_{j+1} = W_j X_j$. One obtains

$$b^q Y^q \geq \sum_{j=0}^{b-1} W_j^q X_j^q - 2(1-q)\sum_{i<j} W_i^{q/2} W_j^{q/2} X_i^{q/2} X_j^{q/2}.$$

Taking expectations yields

$$b^q E(Y^q) \geq bE(W^q)E(X^q) - b(b-1)(1-q)E^2(W^{q/2})E^2(X^{q/2}).$$

By returning to our initial notations,

$$E(Y_n^q) \geq b^{1-h}E(W^q)E(Y_{n-1}^q) - b^{1-q}(b-1)(1-q)E^2(W^{q/2})E^2(Y_{n-1}^{q/2}).$$

Taking account of $E(Y_n^q) \leq E(Y_{n-1}^q)$ (inequality of super-martingales),

$$E(Y_n^q)[1 - b^{1-h}E(W^q)] \geq -b^{1-q}(b-1)(1-q)E^2(W^{q/2})E^2(Y_{n-1}^{q/2})$$

hence

$$E(Y_n^q)(b^{\varphi(q)} - 1) \leq b^{1-q}(b-1)(1-q)E^2(Y_{n-1}^{q/2}),$$

and, by letting $q \to 1$, we get

$$D \log b \leq (b-1)E^2(Y_{n-1}^{1/2}).$$

Now the r.v. $Y_n^{1/2}$ are equi-integrable, since $E(Y_n) = 1$. As they converge a.s. towards Y_∞, one has $E(Y_\infty^{1/2}) = \lim_{n \to \infty} E(Y_n^{1/2})$ (see for example Meyer 1966, II.21), so that $E(Y_\infty^{1/2}) \neq 0$. This implies (β), which concludes the demonstration of Theorem 1.

3. Proof of Theorem 2

First, suppose that (3) has a positive solution Z such that $0 < E(Z^q) < \infty$, with given $q > 1$. Because the function x^q is super-additive, one has

$$b^q Z^q \geq \sum_{j=0}^{b-1} (W_j Z_j)^q,$$

and the strict inequality holds with with positive probability, so that

$$b^q E(Z^q) > bE(W^q)E(Z^q),$$

that is $E(W^q) < b^{q-1}$.

Conversely, suppose that $E(W^q) < b^{q-1}$, so that $\varphi(q) < 0$, and let k be an integer with $k < q \leq k+1$. As the function $x^{q/(k+1)}$ is sub-additive, one has, for $x_j \geq 0$ ($j = 1, 2, \dots, b$),

$$\begin{aligned}(x_1 + x_2 + \dots + x_b)^q &\leq (x_1^{q/k+1} + \dots + x_b^{q/k+1})^{k+1} \\ &= x_1^q + \dots + x_b^q + \sum \gamma_{\alpha_1, \dots, \alpha_b} (x_1^{\alpha_1} \dots x_b^{\alpha_b})^{q/(k+1)}.\end{aligned}$$

In the last sum, the exponents of x_j do not exceed k, the coefficients are positive, and $\sum \gamma_{\alpha_1, \alpha_2, \dots, \alpha_b} = b^{k+1} - b$.

Reconsider formula (2) in the form (11), and remark that $E(U^{q/(k+1)}) \leq E(U)$ if $U \geq 0$ and that $\prod_j E(U^{\alpha_j}) \leq E(U)E(U^k)$ if the α_j are non-negative integers such that $\sum \alpha_j = k+1$, and at least two of the α_j be different from 0. We obtain

$$b^q E(Y^q) \leq bE(W^q)E(X^q) + (b^{k+1} - b)[E(W^k)E(X^k)]^{q/k}.$$

Hence

$$E(Y_n^q) \leq b^{1-q}E(W^q)E(Y_{n-1}^q) + b[E(W^k)E(Y_{n-1}^k)]^{q/k}.$$

Taking into account the sub-martingale inequality $E(Y_n^q) \geq E(Y_{n-1}^q)$

$$E(Y_n^q)(1 - b^{1-h}E(W^q)) \leq b[E(W^k)E(Y_n^k)]^{q/k}.$$

Letting n go to infinity, one can see that

$$E(Y_\infty^k) < \infty \Rightarrow E(Y_\infty^q) < \infty.$$

This establishes the desired result for $1 < q \leq 2$. Now assume that $q > 2$. As the hypothesis $\varphi(q) < 0$ implies $\varphi(l) < 0$ for all integer $l \leq q$, one also has

$$E(Y_\infty^{l-1}) < \infty \Rightarrow E(Y_\infty^l) < \infty$$

for $l = 2, \ldots, k$. It results from the above implications that $E(Y_\infty^q) < \infty$. This concludes the proof of Theorem 2.

4. Proof of Theorem 3

Part 1. According to Theorem 2, (α_1) implies $E(W^q) < b^{q-1}$ for all $q > 1$. This implies (β_1). Conversely, if (β_1) holds, condition (δ) of Theorem 1 is satisfied, hence $E(Y_\infty^q) > 0$. Furthermore $E(W^q) \leq b^q$ i.e. $\varphi(q) \leq 1$ for all $q > 0$. Since $\varphi(1) = 0$ and φ is convex, this means that $\varphi(q) < 0$ for all $q > 1$, that is $E(W^q) < b^{q-1}$, or $\varphi \equiv 0$, i.e. $W \equiv 1$. Hence $E(Y_\infty^q) < \infty$ for all $q > 0$. Therefore (α_1) $\Leftrightarrow$ (β_1).

Part 2. (β_1). Hence (by Theorem 1) $E(Y_\infty) = 1$. On the other hand, there exists an $\varepsilon > 0$ such that $\varphi(q) < \log_b(1 - \varepsilon)$ for $q \geq 2$. That is,

$$E(W^q) \leq (1 - \varepsilon)b^{q-1} \quad (q \geq 2). \tag{12}$$

Consider formula (3), with $Z = Y_\infty$, and let h be an integer ≥ 2. One has

$$b^q Z^q = \left(\sum_{j=0}^{b-1} W_j Z_j \right)^q,$$

from where

$$b^q E(Z^q) = bE(W^q)E(Z^q) + \sum_{\substack{q_1 + \ldots + q_b = q \\ q \le q-1}} \frac{q!}{q_1! \ldots q_b!} \prod_{j=1}^{b} E(W^{q_j}) \prod_{j=1}^{b} E(Z^{q_j}). \tag{13}$$

(13) together with (12) gives

$$\varepsilon b^q E(Z^q) \le \sum_{\substack{q_1 + \ldots + q_b = q \\ q \le q-1}} \frac{q!}{q_1! \ldots q_b!} \prod E(W^{q_j}) \prod E(Z^{q_j}), \tag{14}$$

hence

$$E(Z^q) \le \frac{1}{\varepsilon} \sum_{\substack{q_1 + \ldots + q_b = q \\ q \le q-1}} \frac{q!}{q_1! \ldots q_b!} \prod E(Z^{q_j}).$$

Lemma D. *For all* $\alpha > 0$, *one has*

$$\sum_{\substack{q_1 + \ldots + q_b = q \\ q \le q-1}} (q_1! \ldots q_b!)^\alpha = o((q!)^\alpha) \quad (q \to \infty).$$

The proof follows immediately for $b = 2$, and continues through recurrence over b.

$c > 0$ being given, write $A_q = \sup_{l<q} (E(Z^l)/(l\,!)^{1+c})^{1/l}$. Then

$$A_{q+1}^q \le \sup\left(\frac{1}{\varepsilon} \frac{\sum (q_1! \ldots q_b!)^c}{(q!)^\alpha} A_q^q, A_q^q \right).$$

According to Lemma D, the sequence A_q is bounded, therefore

$$E(Z^q) \le A(c)^q (q!)^{1+c}, \text{ with } A(c) < \infty.$$

As a consequence,

$$\lim\ \sup_{q \to \infty} \frac{\log E(Y_\infty^q)}{q \log q} \le 1. \tag{15}$$

If we suppose that $\|W\|_\infty = \gamma < b$, (14) gives

$$E(Z^q) \le \frac{1}{\varepsilon}\left(\frac{\gamma}{b}\right)^q \sum \frac{q!}{q_1!\ldots q_b!} \prod E(Z^{q_i}).$$

Set $B_q = \sup_{l<q}(E(Z^l)/l!)^{1/l}$.

As the number of terms in the sum Σ does not exceed q^b, one obtains

$$B_{q+1}^q \le \sup\left(\frac{1}{\varepsilon}\left(\frac{\gamma}{b}\right)^q q^b B_q^q, B_q^q\right)$$

which means that the sequence B_q is bounded. It results that $E(e^{tZ}) < \infty$ for all small enough $t > 0$.

Now set $e^{\chi(t)} = E(e^{tZ})$. Formula (3) then becomes

$$e^{\chi(bt)} = E^b(e^{\chi(Wt)}). \tag{16}$$

The hypothesis $\|W\|_\infty = \gamma < b$ implies $\chi(bt) \le b\chi(\gamma t)$, hence $\chi((b/\gamma)^n) = O(b^n)$ for $n \to \infty$. Setting $(b/\gamma)^K = b$, one has

$$\chi(t) = O(t^K) \quad (t \to \infty). \tag{17}$$

It is an easy exercise to show that (17) is equivalent to the existence of a positive real B such that

$$E(Z^q) \le B^q (q!)^{1-(1/K)}.$$

However, $1 - (1/K) = \log_b \gamma$. Hence one has

$$\lim\ \sup_{q\to\infty} \frac{\log E(Y^q)}{q \log q} \le \log_b \gamma. \tag{18}$$

Let us now choose $1 < \gamma_1 < \|W\|_\infty$ (the case $\|W\|_\infty = 1$ is straightforward). There exists an $\varepsilon > 0$ such that $E(W^q) \ge \varepsilon\gamma_1^q$. Reexamine formula (13). Since

$$\sum_{\substack{q_1+\ldots+q_b=q \\ q_i \le q-1}} \frac{q!}{q_1!\ldots q_b!} = b^q - b,$$

one has

$$E(Z^q) \geq \frac{b^q - b}{b^q} \ \inf \prod_{j=1}^{b} E(W^{q_j}) E(Z^{q_j})$$

$$\geq \left| \frac{1}{2} \right| \varepsilon^b \gamma_1^q \ \inf \prod_{j=1}^{b} E(Z^{q_j})$$

the infimum being taken over all b-tuples $(q_1, q_2, \dots, q_b)$ such that $q_1 + q_2 + \cdots + q_b = q$ and $\sup_j q_j \leq q - 1$. Let q be a multiple of b.

The infimum is then $E^b(Z^q/b)$. Hence

$$\frac{\log E(Z^{bq})}{bq} \geq \frac{\log E(Z^q)}{q} + \log \gamma_1 + O\left(\frac{1}{q}\right),$$

with the consequence

$$\log E(Z^q) \geq \eta q \log q + O(q), \quad \eta = \log_b \gamma_1 \tag{19}$$

from where

$$\liminf_{q \to \infty} \frac{\log E(Y_\infty^q)}{q \log q} \geq \log_b \gamma_1 \tag{20}$$

(15), (18) and (20) lead to

$$\lim_{q \to \infty} \frac{\log E(Y_\infty^q)}{q \log q} = \log_b \|W\|_\infty.$$

This terminates the proof of Theorem 3.

Remark. In the case $0 < P(W = b) < 1/b$, one has $\gamma_1 = b$ in (19), from which follows that $E(e^{tZ}) = \infty$ for large enough $t > 0$.

5. Proof of Theorem 4

Let Ω be the space on which the random variables $W(j_1, j_2, \dots, j_n)$ are defined. On the product space $\Omega \times [0, 1]$, consider the probability Q defined by

$$Q(A) = E\left(\int 1_A d\mu\right).$$

Let $X_n = \sum_{j_1, \dots, j_n} W(j_1, \dots, j_n) 1_{I(j_1, \dots, j_n)}$. One has then

$$\mu_n = b^{-n} \prod_{1 \le j \le n} X_n$$

(with an evident abuse of notation).

Write $\mu = \mu_n \nu_n$; here ν_n is a measure whose restriction to each interval of the n-th stage is defined in an analogous fashion as for μ.

Observe that the variables $\nu_n(I(j_1, \dots, j_n))$ have the same distribution as Y_∞ and that, for fixed n, they are mutually independent. Moreover, the variables $\nu_n(I(j_1, \dots, j_n))$ and $W(k_1, \dots, k_n)$ are independent as long as the interval $I(k_1, \dots, k_p)$ is not strictly contained in the interval $I(j_1, \dots, j_n)$.

It is convenient to consider the random function

$$T_n = \sum_{j_1, \dots, j_n} b^n \nu_n(I(j_1, \dots, j_n)) 1_{I(j_1, \dots, j_n)}.$$

If u is a function defined on $[0, 1]$, constant on the intervals of the n-th stage, one has

$$\int u d\mu = \int_0^1 u(x) \mu_n(x) T_n(x) dx. \tag{21}$$

The theorem results from the two following Lemmas.

Lemma E. *If* $E(W \log_b W) < 1$, *then almost surely* μ*-almost everywhere* $(1/n) \log \mu_n$ *tends towards* $E(W \log W)$ *when* $n \to +\infty$.

Proof. We will show that

$$\sum_{n \ge 1} Q(\{X_n > e^{n-1}\}) < \infty \tag{22}$$

holds and that the series

$$\sum_{n \geq 1} \frac{1}{n} \{ \log \ \inf (X_n, e^{n-1}) - E[W \log(\inf (W, e^{n-1}))]\} \tag{23}$$

converges Q-almost surely. Therefore, one will have Q-almost surely $X_n \leq e^{j-1}$ beyond a certain rank and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log \inf(X_j, e^{n-1}) = E(W \log W),$$

from which the lemma follows.

Let us begin by evaluating $Q([X_n > e^{n-1}])$. One has

$$Q([X_n > e^{n-1}]) = E\left[\int 1_{[X_n > e^{n-1}]} d\mu\right].$$

Taking account of (21) and the properties of independence of variables, we obtain

$$\begin{aligned} Q(\{X_n > e^{n-1}\}) &= E\int_0^1 1_{\{X_n(x) > e^{n-1}\}} \mu_n(x) T_n(x) dx \\ &= \int_0^1 E(X_n(x) 1_{\{X_n(x) > e^{n-1}\}}) E(\mu_{n-1}(x)) E(T_n(x)) dx \\ &= E(W 1_{\{W > e^{n-1}\}}), \end{aligned}$$

from where we have

$$\sum_{n \geq 1} Q(\{X_n > e^{n-1}\}) \leq E\left\{W \sum_{n \geq 1} 1_{\{W > e^{n-1}\}}\right\} \leq E(W(1 + \log^+ W)),$$

which proves (22).

For ease of writing, set $X'_n = \log \inf(X_n, e^{n-1})$. We will calculate $E_Q(X'_n | X_1, \ldots, X_{n-1})$. Let u be a bounded Borel function from $\mathbb{R}^{n-1}$ into $\mathbb{R}$. One has

$$\int u(X_1, \dots, X_{n-1})X'_n dQ$$
$$= E\int_0^1 u(X_1(x), \dots, X_{n-1}(x))X'_n(x)\mu_n(x)T_n(x)dx$$
$$= \int_0^1 E[u(X_1(x), \dots, X_{n-1}(x))\mu_{n-1}(x)]E[X'_n(x)X_n(x)]dx$$
$$= E[W \log \inf(W, e^{n-1})]\int_0^1 E[u(X_1(x), \dots, X_{n-1}(x))\mu_{n-1}(x)T_{n-1}(x)]dx.$$

This proves that $E_Q(X'_n | X_1, \dots, X_{n-1}) = E[W \log \inf(W, e^{n-1})]$. Therefore

$$\int (X'_n)^2 dQ = E\int_0^1 (X'_n(x))^2 \mu_n(x)T_n(x)dx = E[W(\log \inf(W, e^{n-1}))^2],$$

hence

$$\sum_{n\geq 2} \frac{1}{n^2}\int (X'_n)^2 dQ$$
$$= E\left[W\sum_{n\geq 2}\frac{1}{n^2}(\log \inf(W, e^{n-1}))^2\right]$$
$$\leq E\left[W(\log W)^2 \sum_{n\geq \sup(2, 1+\log W)} n^{-2} + W \sum_{2\leq n<1+\log W}\left(\frac{n-1}{n}\right)^2\right]$$
$$\leq E\left[W\left(\log^+ W + \frac{(\log W)^2}{\sup(1, \log W)}\right)\right].$$

The theorem on the convergence of L^2-martingales gives (23).

Lemma F. *Suppose that $E(W \log_b W) < 1$ and that $E(Y_\infty \log Y_\infty) < \infty$. Then almost surely μ-almost everywhere $(1/n) \log \nu_n(I_n(x))$ tends towards $-\log b$.*

Proof. One has

$$\int T_n^{-1/2} dq = E\int_0^1 \mu_n(x)(T_n(x))^{1/2}dx = E(Y_\infty^{1/2}),$$

from where

$$\int\left(\sum_{n\geq 1}\frac{1}{n^2}T_n^{-1/2}\right)dQ<\infty.$$

As a consequence, Q-almost surely, $T_n^{-1/2}\leq n^2$ holds above a certain n, hence $\liminf_{n\to\infty}(1/n)\log T_n\geq 0$. Up to now, we have not used the second hypothesis. Now let us show that, Q almost surely, one has lim sup adjust(u 2)$_{n\to\infty}(1/n)\log T_n\leq 0$. Take a number $\alpha>1$. Using (21),

$$E\int 1_{\{T_n>\alpha^n\}}d\mu=E(Y_\infty 1_{\{Y_\infty>\alpha^n\}}).$$

Then

$$\sum_{n\geq 1}Q(\{T_n>\alpha^n\})=E\left(Y_\infty\sum_{n\geq 1}1_{\{Y_\infty>\alpha^n\}}\right)\leq E(Y_\infty\log_\alpha^+Y_\infty)$$

follows, proving that, for all $\alpha>1$, it is Q-almost sure that lim sup adjust(u 2)$_{n\to\infty}(1/n)\log T_n\leq\log\alpha$.

The desired result follows. As $\nu_n(I_n(x))=b^{-n}T_n(x)$, the lemma is proven. To prove the corollary, use Billingsley 1967, p. 136-145.

Remark. Under the only hypothesis $E(W\log_b W)<1$, one obtains that almost surely every Borel set of dimension $<D$ is of vanishing μ-measure.

&&&&& POST-PUBLICATION APPENDIX &&&&&

AN ELEMENTARY EXPLICIT SOLUTION OF MANDELBROT'S FUNCTIONAL EQUATION $bY=\sum_{j=1}^{b}W_jY_j$, BY J. PEYRIÈRE (1998)

Mandelbrot's equation starts with an integer base $b>1$ and a random variable $W\geq 0$ satisfying $EW=1$ and $E(W\log W)<\log b$. The unknown is a random variable Y satisfying $EY=1$. This appendix proposes to show that an elementary explicit solution exists in the special case when $0<W<b$ and $\Pr\{W<w\}=(w/b)^\alpha$ for $w<b$, with $\alpha=1/(b-1)$. In that case, the gen-

erating function of Y, written as $g(t) = E(e^{-tY})$, verifies the functional equation

$$g(t) = \left(\alpha \int_0^1 g(tx)x^{\alpha-1}dx\right)^b,$$

with $g(0) = -g'(0) = 1$. Defining $h(t) = g(t)^{(1/b)}$, one has $h(t) = \alpha\int_0^1 h(tx)^b x^{\alpha-1}dx$, $h(0) = 1$ and $h'(0) = -1/b$. Differentiating under the integral sign and integrating by parts yields the differential equation

$$(b-1)h'(t) = \frac{h(t)^b - h(t)}{t},$$

from which it follows that $\dfrac{1-h(t)^{1-b}}{t}$ is a constant. The final result is

$$g(t) = \left(1 + \frac{b-1}{b}t\right)^{\frac{-b}{b-1}}.$$

That is, the random variable $Yb/(b-1)$ follows the gamma distribution of parameter $b/(b-1) = \alpha + 1$.

&&&&&&&&&&&& ANNOTATIONS &&&&&&&&&&&&

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Editorial changes. The text was divided into sections and the notation changed to fit the present usage. The original denoted b by c, q by h, q_{crit} by α_0, and c by α.

More recent developments. The decision to translate and reproduce this text was motivated by the desire to correct an injustice: indeed, the results of this paper are often credited to later authors. Those results, with additional ones, were restated in Kahane 1987b, which summarizes Kahane 1987a, and other papers written in French. In particular, the corollary of Theorem 4 is shown to hold under wider and more natural conditions. There are further extensions in Kahane 1989, 1991a.