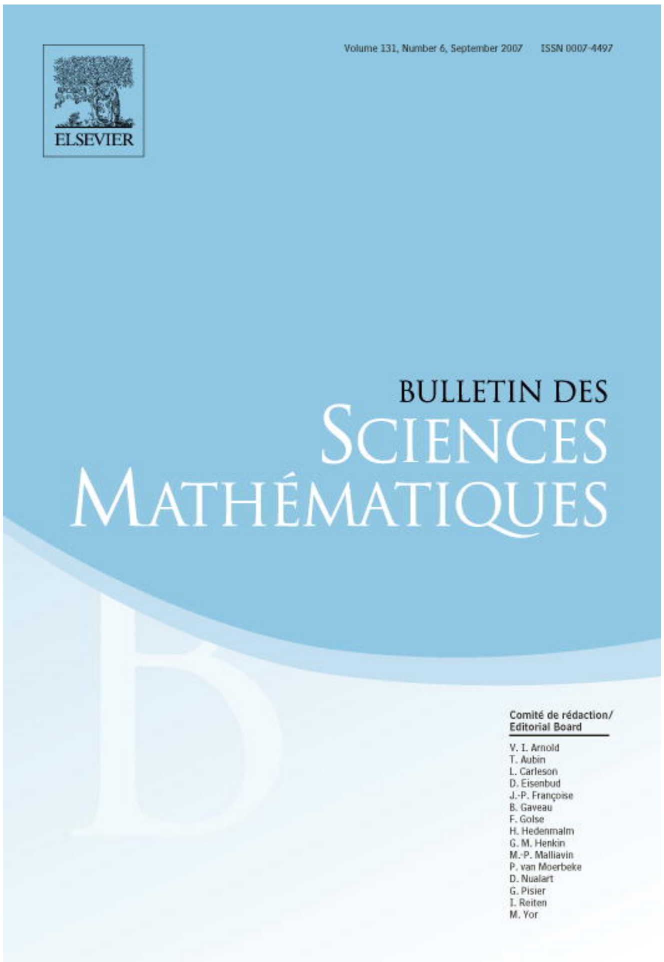
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Periodic polynomial of trace maps

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Abstract

Let σ be an endomorphism of the free group on two generators and Φ_σ the trace map associated with σ. A polynomial P is said to be *periodic* for σ if, for some positive integer n, it is invariant under Φ_σ^n, i.e., $P \circ \Phi_\sigma^n = P$. In this note we study the structure of the ring of periodic polynomials for σ.

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Keywords: Free groups; Fricke characters; Trace maps

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1. Introduction

Iteration of polynomial maps from $\mathbb{C}^3$ to $\mathbb{C}^3$ appears, for instance, when studying discrete Schrödinger operators with potentials given by substitutive sequences (see, for instance, [1–6,9,10]). These polynomial maps, usually called *trace maps*, are associated with endomorphisms of the free group of rank 2.

If for such a map Φ there exists a non-constant polynomial P such that $P \circ \Phi = P$, then the space $\mathbb{C}^3$ is foliated and each sheet is invariant under Φ. This means that this dynamical system operates on spaces of dimension 2 instead of 3. This is one of the motivations to investigate the existence of such invariant polynomials.

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Before stating our results, we recall some definitions that can be found in [14,15].

1.1. Fricke characters

Let $\mathcal{A} = \{a, b\}$ be a two-letter alphabet. Let $\mathcal{A}^*$ and F be the free monoid and the free group generated by $\mathcal{A}$. We denote by $\operatorname{End} F$ and $\operatorname{Aut} F$ the monoid of endomorphisms and the group of automorphisms of F. The composition $\sigma \circ \tau$ of two morphisms will be simply denoted by $\sigma\tau$.

We identify $\sigma \in \operatorname{End} F$ with the pair $(\sigma(a), \sigma(b))$. When both $\sigma(a)$ and $\sigma(b)$ are in $\mathcal{A}^*$, the morphism σ is called a substitution on the alphabet $\mathcal{A}$.

By abelianization, an element $\sigma \in \operatorname{End} F$ defines a $\mathbb{Z}$-linear endomorphism M_σ of $\mathbb{Z}^2$. In other terms, M_σ is a 2×2-matrix with integer entries. More precisely, the entry of index $(i, j) \in \mathcal{A}^2$ is the sum of the exponents of the letter i in the word $\sigma(j)$. One has $M_{\sigma_1\sigma_2} = M_{\sigma_1} M_{\sigma_2}$.

Denote by $SL(2, \mathbb{C})$ the set of 2×2 matrices with complex coefficients and determinant 1. Denote by $\operatorname{Hom}(F, SL(2, \mathbb{C}))$ the set of group homomorphisms from F to $SL(2, \mathbb{C})$.

For any word $w \in F$, there exists (see [7,14]) a unique polynomial $P_w \in \mathbb{Z}[x, y, z]$ such that, for any $\varphi \in \operatorname{Hom}(F, SL(2, \mathbb{C}))$, one has

$$\operatorname{tr} \varphi(w) = P_w\big(\operatorname{tr} \varphi(a), \operatorname{tr} \varphi(b), \operatorname{tr} \varphi(ab)\big). \tag{1}$$

One has $P_a = x$, $P_b = y$, and $P_{ab} = z$.

The polynomial

$$\lambda(x, y, z) = P_{aba^{-1}b^{-1}} - 2 = x^2 + y^2 + z^2 - xyz - 4 \tag{2}$$

is of particular importance. Indeed, two 2×2 unimodular matrices A and B share an eigendirection if and only if $\lambda(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = 0$. The polynomial λ is irreducible.

1.2. Trace maps

Let $\sigma \in \operatorname{End} F$, the trace map Φ_σ associated with σ is the map from $\mathbb{C}^3$ to $\mathbb{C}^3$ defined to be (see [8,13,14])

$$\Phi_\sigma = \big(P_{\sigma(a)}, P_{\sigma(b)}, P_{\sigma(ab)}\big). \tag{3}$$

Indeed Φ_σ is the unique polynomial map such that, for all $\varphi \in \operatorname{Hom}(F, SL(2, \mathbb{C}))$, one has $(\operatorname{tr} \varphi \circ \sigma(a), \operatorname{tr} \varphi \circ \sigma(b), \operatorname{tr} \varphi \circ \sigma(ab)) = \Phi_\sigma(\operatorname{tr} \phi(a), \operatorname{tr} \phi(b), \operatorname{tr} \phi(ab))$.

It results that, for any $w \in F$ and σ and $\tau \in \operatorname{End} F$, one has $\Phi_{\sigma\tau} = \Phi_\tau \circ \Phi_\sigma$ and $P_{\sigma(w)} = P_w \circ \Phi_\sigma$.

Another important fact is that, for any $\sigma \in \operatorname{End} F$, the polynomial λ divides $\lambda \circ \Phi_\sigma$: one defines the polynomial $Q_\sigma \in \mathbb{Z}[x, y, z]$ by the relation $\lambda \circ \Phi_\sigma = \lambda . Q_\sigma$. For σ and $\tau \in \operatorname{End} F$, one has $Q_{\sigma\tau} = Q_\sigma . Q_\tau \circ \Phi_\sigma$.

1.3. Invariant polynomials

The monoid $\operatorname{End} F$ acts on $\mathbb{C}[x, y, z]$ in the following way: for $\sigma \in \operatorname{End} F$ and $P \in \mathbb{C}[x, y, z]$, one defines $\sigma P = P \circ \Phi_\sigma$. Since, for any $\sigma, \tau \in \operatorname{End} F$, $\Phi_{\sigma\tau} = \Phi_\tau \circ \Phi_\sigma$, one has $\sigma(\tau P) = (\sigma\tau)P$.

A polynomial P such that $\sigma P = P$ is said σ-invariant. A polynomial σ^k-invariant for some k is said to be periodic for σ.

The set $\mathcal{R}_\sigma$ of σ-periodic polynomials is a sub-ring of $\mathbb{C}[x, y, z]$.

1.4. Main results

Theorem 1. *If σ is a primitive invertible substitution, then $\mathcal{R}_\sigma = \mathbb{C}[\lambda]$.*

Theorem 2. *If $\sigma \in \operatorname{Aut} F$, then*

(1) *If none of the eigenvalues of M_σ is a root of 1, then $\mathcal{R}_\sigma = \mathbb{C}[\lambda]$;*
(2) *If both eigenvalues of M_σ are roots of 1 and M_σ is similar to a diagonal matrix, then $\mathcal{R}_\sigma = \mathbb{C}[x, y, z]$;*
(3) *If both eigenvalues of M_σ are roots of 1, but M_σ is not similar to a diagonal matrix, then there exists $P \in \mathbb{C}[x, y, z]$ such that $\mathcal{R}_\sigma = \mathbb{C}[\lambda, P]$.*

Theorem 3. *Let $\sigma \in \operatorname{End} F$. If $Q_\sigma \not\equiv 0$ or 1 and none of the eigenvalues of M_σ is a root of 1. Moreover, suppose that M_σ is invertible or σ is Nielsen reduced. Then σ has no periodic polynomials, except constant ones.*

This article is organized as follows. In Section 2, we give some preliminaries, list some known results, and prove some lemmas. In Section 3, we determine the ring $\mathcal{R}_\sigma$ for any invertible trace map. In Section 4, we discuss the case of non-invertible trace maps.

2. Preliminaries

First we recall some known results which can be found in [7,8,12,14] and [15].

Theorem A1. *The group $\operatorname{Aut} F$ is generated by the three following morphisms: $\alpha = (ab, a)$, $\pi = (b, a)$, and $\pi^* = (a, b^{-1})$.*

2.1. Trace maps

Theorem A2. *With the above definitions and notations, we have the following facts.*

(1) *For any $\sigma \in \operatorname{End} F$, there exists a polynomial $Q_\sigma \in \mathbb{Z}[x, y, z]$ such that $\lambda \circ \Phi_\sigma = \lambda \cdot Q_\sigma$, where $\lambda(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$;*
(2) *If Q_σ is constant, then $Q_\sigma = 0$ or $Q_\sigma = 1$;*
(3) *$Q_\sigma \equiv 1$ if and only if $\sigma \in \operatorname{Aut} F$;*
(4) *$Q_\sigma \equiv 0$ if and only if σ is not one-to-one;*
(5) *If $\sigma, \tau \in \operatorname{Aut} F$, then $\Phi_\sigma = \Phi_\tau$ if and only if $M_\sigma = \pm M_\tau$.*

2.2. The map Π

Let Ω be the variety of zeros of λ

$$\Omega = \{(x, y, z) \in \mathbb{C}^3\colon\ \lambda(x, y, z) = 0\}, \tag{4}$$

and Π the mapping from $\mathbb{C}^2$ to $\mathbb{C}^3$ defined by the formula

$$\Pi(\alpha, \beta) = (2\cos\alpha, 2\cos\beta, 2\cos(\alpha + \beta)).$$

Then one has $\Pi(\mathbb{C}^2) = \Omega$. Moreover, one has [14],

Theorem A3. *For any* $\sigma \in \operatorname{End} F$, $\Phi_\sigma \circ \Pi = \Pi \circ {}^{\mathrm t}M_\sigma$.

2.3. Analytic invariant function

A function $T : \mathbb{C}^2 \to \mathbb{C}$ is called an analytic invariant function for the linear transformation $M : \mathbb{C}^2 \to \mathbb{C}^2$ if T is entire and $T \circ M = T$.

Lemma 4. *Let* $M = \left[\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right]$. *Then there exists a non-constant analytic function invariant for M if and only if there exist non-negative integers m and n with $m + n \neq 0$ such that* $\delta_1^m \delta_2^n = 1$.

Proof. If there are m and n in $\mathbb{N}$ such that $m + n \neq 0$ and $\delta_1^m \delta_2^n = 1$, then $x^m y^n$ is an invariant polynomial for M.

Now, suppose that $T(x, y)$ is a non-constant analytic function invariant under M. So, there exists a non-zero term $cx^m y^n$ in the Taylor expansion of T at the origin. The corresponding term for $T \circ M$ is $c\delta_1^m \delta_2^n x^m y^n$. So one has $\delta_1^m \delta_2^n = 1$. □

Lemma 5. *Let* $\delta \in \mathbb{R}$ *and* $M = \left[\begin{smallmatrix} \delta & 0 \\ 1 & \delta \end{smallmatrix}\right]$. *Then there exists a non-constant analytic function invariant under M if and only if* $\delta = \pm 1$.

Proof. The homogeneous components of the Taylor expansion of an invariant function are also invariant. So we just have to look for the invariant homogeneous polynomials of positive degree n. Let P be such a polynomial. It can be written as $P(x, y) = x^n p(y/x)$, where p is a univariate polynomial of degree less than or equal to n. The invariance of P means $\delta^n p(\frac{1}{\delta} + t) = p(t)$. By looking at the term of highest degree, one sees that one should have $\delta = \pm 1$. Obviously, if $\delta = \pm 1$, then x^2 is invariant. □

Lemma 6. *If* $M = \left[\begin{smallmatrix} 1 & 0 \\ \alpha & \beta \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, *then the M-invariant functions are those which can be written as $(x, y) \mapsto f(x)$, where f is an entire function.*

Proof. As in the preceding proof, we look for invariant polynomials of the form $x^n p(y/x)$. One must have $p(\alpha + \beta t) = p(t)$, which implies that p is constant. □

2.4. A key lemma

Almost all our discussions are based on the following lemma.

Lemma 7. *Let* $\sigma \in \operatorname{End} F$ *and* Φ_σ *be the associated trace map. If there is a non-constant polynomial P periodic for σ, then one of the following alternatives holds*:

(1) *There exists* $c \in \mathbb{C}$, *such that* $\lambda | (P - c)$;
(2) *At least one of the eigenvalues of the matrix* M_σ *belongs to the set*

$$B = \left\{ \pm 1, \pm i, \frac{1 \pm \sqrt{3}i}{2}, -\frac{1 \pm \sqrt{3}i}{2} \right\}. \tag{5}$$

Proof. Consider first the case when P is σ-invariant, i.e., $\sigma P = P$. By Theorem A3,

$$P \circ \Pi \circ {}^{\mathrm t}M_\sigma = P \circ \Phi_\sigma \circ \Pi = (\sigma P) \circ \Pi = P \circ \Pi. \tag{6}$$

If there exists a constant $c \in \mathbb{C}$ such that $P \circ \Pi \equiv c$, then we have $(P-c)|_{\Omega} \equiv 0$, and $\lambda | (P-c)$.

Suppose there is no such c. Then the analytic function $P \circ \Pi$ is non-constant and invariant under the linear map ${}^{\mathrm{t}}M_\sigma$.

We consider first the case when the matrix M_σ is diagonalizable, that is, there is an invertible matrix V, such that

$$M_1 = V^{-1}\,{}^{\mathrm{t}}M_\sigma V = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix},$$

thus

$$P \circ \Pi \circ V \circ M_1 = P \circ \Pi \circ {}^{\mathrm{t}}M_\sigma \circ V = P \circ \Pi \circ V.$$

Since V is invertible, $V(\mathbb{C}^2) = \mathbb{C}^2$ and $P \circ \Pi \circ V$ still is non-constant. So, $P \circ \Pi \circ V$ is a non-constant analytic function invariant under M_1. Due to Lemma 4, there exist non-negative m and n such that $m+n \neq 0$ and $\delta_1^m \delta_2^n = 1$, where δ_1 and δ_2 are the eigenvalues of M_σ. We may suppose that $m \geqslant n$.

Now we only need to prove that δ_1 or δ_2 belong to the set B. We distinguish three cases according to the value of $\det M_\sigma = \delta_1\delta_2 \in \mathbb{Z}$.

Case 1: $\det M_\sigma \notin \{0, \pm 1\}$.

First, we show that $m > n$. In fact, if $m = n$, then from $1 = \delta_1^m \delta_2^n = (\det M_\sigma)^n$, we get $n = 0$. This contradicts $m + n \neq 0$.

Second, we claim that $n = 0$. If $n \neq 0$, then $k = (\delta_1\delta_2)^n$ is an integer such that $|k| > 1$, and

$$k\delta_1^{m-n} - 1 = 0.$$

On the other hand, δ_1 and δ_2 are the roots of the following monic polynomial with integer coefficients

$$x^2 - (\operatorname{tr} M_\sigma)x + \det M_\sigma = 0.$$

Newton's formulae on symmetric polynomials show that δ_1^{m-n} and δ_2^{m-n} are algebraic integers of degree less than or equal to 2, which leads to a contradiction.

Since $m > n = 0$, we have $\delta_1^m = 1$. If δ_1 were not real, then $\det M_\sigma = \delta_1\delta_2 = |\delta_1|^2$ would be equal to 1, contrary to the hypothesis. So, in this case, one of the eigenvalues of M_σ is equal to ± 1.

Case 2: $\det M_\sigma = 0$.

We may suppose $\delta_2 = 0$. Since, $\delta_1^m \delta_2^n = 1$, we see that $n = 0$ and $\delta_1^m = 1$. But, as $\delta_1 = \operatorname{tr} M_\sigma$ is an integer, $\delta_1 = \pm 1$. This implies again that one eigenvalue of M_σ is equal to ± 1.

Case 3: $\det M_\sigma = \pm 1$.

In this case, δ_1 and δ_2 satisfy one of the following equations

$$x^2 - \alpha x + 1 = 0, \quad \alpha \in \mathbb{Z}, \tag{7}$$

$$x^2 - \beta x - 1 = 0, \quad \beta \in \mathbb{Z}. \tag{8}$$

It is easy to see that the roots of (7) and (8) are irrational real numbers if and only if $\alpha \neq 0, \pm 1, \pm 2$ and $\beta \neq 0$.

Now, suppose δ_1 and δ_2 are irrational real numbers. Then ${}^{\mathrm{t}}M_\sigma$ has two real eigenvectors. Let l_1 and l_2 be the real subspaces which they generate. Moreover, since $|\delta_1| > 1 > |\delta_2|$ or $|\delta_1| < 1 < |\delta_2|$, one has

$$\lim_{k\to\infty} ({}^{\mathrm{t}}M_\sigma)^k(p) = (0,0), \quad \text{or} \quad \lim_{k\to-\infty} ({}^{\mathrm{t}}M_\sigma)^k(p) = (0,0), \quad \forall p \in l_1 \cup l_2. \tag{9}$$

Consider the variety

$$E = \{(x, y, z) \in \mathbb{C}^3\colon\ P(x, y, z) = P(2, 2, 2)\}.$$

Since P is σ-invariant, E is a closed invariant set under Φ_σ. Since by (6), $P \circ \Pi \circ {}^{\mathrm{t}}M_\sigma^k = P \circ \Pi$, we get from (9) and the continuity of $P \circ \Pi$,

$$P \circ \Pi(l_1) = P \circ \Pi(l_2) = \{P \circ \Pi(0, 0)\} = \{P(2, 2, 2)\}.$$

So $\Pi(l_1 \cup l_2) \subset E$.

Define $\Pi_1 : \mathbb{R}^2 \to \mathbb{T}^2$ by $\Pi_1(\alpha, \beta) = (\alpha \bmod 2\pi, \beta \bmod 2\pi)$. Since the slope of l_1 is irrational, the set $\Pi_1(l_1)$ is dense in $\mathbb{T}^2$. This implies that $\Pi(l_1)$ is dense in $\Pi(\mathbb{R}^2)$. It follows that

$$P \circ \Pi(\mathbb{R}^2) = P \circ \Pi(l_1) = \{P \circ \Pi(0, 0)\} = \{P(2, 2, 2)\}.$$

Since the Zariski closure of $\Pi(\mathbb{R}^2)$ is Ω, this contradicts the hypothesis that $P \circ \Pi$ is not constant. Consequently $\alpha \in \{0, \pm 1, \pm 2\}$ or $\beta = 0$, depending on the sign of $\det M_1$.

The roots of (7) and (8), if $\alpha = 0, \pm 1, \pm 2$ and $\beta = 0$, may assume only the values ± 1, $\pm i$, or $\pm \frac{1 \pm \sqrt{3}i}{2}$.

Now one has to handle the case when $P \circ \Pi \not\equiv c$ and M_σ is not diagonalizable. This time $V^{-1}\,{}^{\mathrm{t}}M_\sigma V = \left[\begin{smallmatrix} \delta & 0 \\ 1 & \delta \end{smallmatrix}\right]$. Then Lemma 5 and a discussion similar to the previous one yield $\delta = \pm 1$.

Now, if P is σ-periodic and non-constant, it is σ^k- invariant for some k. So, if P is not of the form $c + \lambda P_1$, the matrix M_σ must have an eigenvalue which is a root of 1. But, as the elements of B are the only algebraic integers of degree less than or equal to 2 which are roots of 1, one see that then at least one of the eigenvalues of the matrix M_σ is in B. □

3. The invertible case

We determine $\mathcal{R}_\sigma$ for any automorphism in Aut F in this section.

Proposition 8. *Let $\sigma \in \operatorname{Aut} F$. If none of the eigenvalues of the matrix M_σ is in B, then $\mathcal{R}_\sigma = \mathbb{C}[\lambda]$.*

Proof. Let P be a non-constant periodic polynomial for σ. Then, due to Lemma 7, there exists $c_0 \in \mathbb{C}$ such that $\lambda | (P - c_0)$, say $P = \lambda P_1 + c_0$. Then

$$\lambda P_1 + c_0 = P = \sigma P = \sigma(P_1 \cdot \lambda) + c_0 = (\sigma P_1) \cdot \lambda + c_0,$$

which means $\sigma P_1 = P_1$. If P_1 is not constant, then, as above, one can write $P_1 = \lambda P_2 + c_1$ where $c_1 \in \mathbb{C}$ and P_2 is σ-invariant. And so on ... This process comes to an end in a finite number, say n, of steps, since the degrees of the polynomials P_j decrease. Then one has

$$P = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n. \qquad \square$$

The next lemma deals with the primitivity of a substitution. Note that if σ is a substitution then M_σ is a nonnegative matrix. A substitution σ is said to be *primitive*, if the matrix M_σ is primitive, that is, if there exists $N \in \mathbb{N}$ such that all the entries of the matrix M_σ^N are positive.

Lemma 9. *Let σ is a substitution, if both eigenvalues of M_σ are roots of* 1*, then σ is non-primitive.*

Proof. Suppose the eigenvalues of M_σ are roots of 1, then there exists $k \in \mathbb{N}$ such that both eigenvalues of M_σ^k are 1, let

$$M_\sigma^k = \begin{bmatrix} m & m' \\ n & n' \end{bmatrix}$$

then

$$m + n' = 2, \quad mn' - m'n = 1, \quad \text{with } m, n, m', n' \geqslant 0.$$

If $(m, n') = (2, 0)$ or $(0, 2)$, then $m'n < 0$, this contradicts $m', n \geqslant 0$. Consequently $m = n' = 1$, so $m' = 0$ or $n = 0$. Hence for any $l \in \mathbb{N}$, M_σ^{lk} is diagonal or triangular and therefore M_σ is not primitive. □

Proof of Theorem 1. Since the substitution σ is invertible, then $\det M_\sigma = \pm 1$. Hence if one eigenvalue of M_σ is in B (defined by (5)), then the other eigenvalue must be in B. Thus the primitivity of σ leads to a contradiction, due to Lemma 9. So by Proposition 8, $\mathcal{R}_\sigma = \mathbb{C}[\lambda]$. □

Proposition 10. *Let $\sigma \in \operatorname{End} F$. Suppose one of the eigenvalues of M_σ is an integer, then there exists $\tau \in \operatorname{Aut} F$ such that $M_{\tau^{-1}\sigma\tau}$ is a upper triangular.*

Proof. Assume $\delta \in \mathbb{Z}$ is an eigenvalue of M_σ, and (m, n) is an integer eigenvector of M_σ corresponding to δ. Without loss of generality, one may assume that m and n are co-prime. Then there exist k and $l \in \mathbb{Z}$ such that $ml - nk = 1$.

Let $V = \left[\begin{smallmatrix} m & k \\ n & l \end{smallmatrix}\right]$, then $\det V = 1$ and $V^{-1} M_\sigma V = \left[\begin{smallmatrix} \delta & p \\ 0 & q \end{smallmatrix}\right]$. By the method of elementary transformations, one can decompose V as product of finitely many matrices in $\{M_\alpha, M_\pi, M_{\pi^*}\}$, where α, π and π^* are defined in Theorem A1. Due to Theorem A1 and Theorem A2(5), there exists $\tau \in \operatorname{Aut} F$ such that $M_\tau = V$. Let $\sigma_1 = \tau^{-1}\sigma\tau$, then

$$M_{\sigma_1} = \begin{bmatrix} \delta & p \\ 0 & q \end{bmatrix}. \qquad \square$$

Proposition 11. *Let $\sigma \in \operatorname{End} F$, $\tau \in \operatorname{Aut} F$, and $\sigma_1 = \tau^{-1}\sigma\tau$, then P is σ-invariant if and only if $\tau^{-1}P$ is σ_1-invariant.*

Proof. Indeed, one has $\tau^{-1}P = \tau^{-1}\sigma P = \tau^{-1}\sigma\tau\tau^{-1}P = \sigma_1(\tau^{-1}P)$. □

3.1. Proof of Theorem 2

Proof. (1) The first assertion is exactly Proposition 8.

(2) Since at least one of the eigenvalues of M_σ is a root of 1, then, by arguing as in the proof of Theorem 1, one shows that there exists $n \in \mathbb{N}$ such that both eigenvalues of M_σ^n are 1. By Proposition 10, there exists $\tau \in \operatorname{Aut} F$ such that $M_{\tau^{-1}\sigma^n\tau}$ is upper triangular. Let $\sigma_1 = \tau^{-1}\sigma^n\tau$, since M_σ is similar to a diagonal matrix, M_{σ_1} is the unit matrix of order 2. Then, due to Theorem A2(5), $\Phi_{\sigma_1}(x, y, z) = (x, y, z)$, which proves the second assertion.

(3) By hypothesis, M_σ is not similar to a diagonal matrix and both its eigenvalues are roots of 1. This implies than its eigenvalues are 1 (or -1). From Proposition 10, there exists $\tau \in \operatorname{Aut} F$ such that $M_{\tau^{-1}\sigma^2\tau}$ is upper triangular. Let

$$\sigma_2 = \tau^{-1}\sigma^2\tau$$

then there is $l \in \mathbb{Z} \setminus \{0\}$ such that

$$M_{\sigma_2} = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}.$$

By Theorem A2(5),

$$\Phi_{\sigma_2}(x,y,z) = \Phi_{(a,a^l b)}(x,y,z) = (x,*,*),$$

from which it follows that the polynomial x, thus any $P \in \mathbb{C}[x]$, is σ_2-invariant.

Let P be σ_2-periodic. By Theorem A3,

$$P \circ \Pi \circ {}^{\mathrm{t}}M_{\sigma_2}^k = P \circ \Pi \quad \text{for some } k.$$

Due to Lemma 6, the analytic function $P \circ \Pi(\alpha,\beta)$ is independent of β.

Now, we are going to show that $P \in \mathbb{C}[x,\lambda]$.

Dividing P by λ as polynomials in z yields

$$P(x,y,z) = P_1(x,y) + P_2(x,y)z + \lambda \cdot P_3(x,y,z).$$

So

$$P \circ \Pi(\alpha,\beta) = P_1(2\cos\alpha, 2\cos\beta) + 2P_2(2\cos\alpha, 2\cos\beta)\cos(\alpha+\beta) = C - D,$$

where

$$C = P_1(2\cos\alpha, 2\cos\beta) + 2P_2(2\cos\alpha, 2\cos\beta)\cos\alpha\cos\beta,$$

$$D = 2P_2(2\cos\alpha, 2\cos\beta)\sin\alpha\sin\beta.$$

Since $D = (P \circ \Pi(\alpha,\beta) - P \circ \Pi(\alpha,-\beta))/2$ and $P \circ \Pi(\alpha,\beta)$ is independent of β, one has $D = 0$, which means $P_2(2\cos\alpha, 2\cos\beta) \equiv 0$. It follows that $P_1(2\cos\alpha, 2\cos\beta)$ is a function of α only, so $P_1 \in \mathbb{C}[x]$, i.e.,

$$P(x,y,z) = P_1(x) + \lambda \cdot P_3(x,y,z).$$

Since P_1 and λ are σ_2-periodic, the polynomial P_3 also is periodic. By iterating this reasoning, one finally gets $P \in \mathbb{C}[x,\lambda]$, i.e., $\mathcal{R}_{\sigma_2} = \mathbb{C}[x,\lambda]$. By Proposition 11,

$$\mathcal{R}_\sigma = \mathbb{C}[\tau^{-1}x, \tau^{-1}\lambda] = \mathbb{C}[\tau^{-1}x, \lambda],$$

from which the third assertion of Theorem 2 follows. □

4. The non-invertible case

In this section, we determine $\mathcal{R}_\sigma$ for some endomorphisms in End F.

We adopt the usual definitions in combinatorial group theory, in particular concerning reduction and cyclic reduction (see for instance [11]).

The following lemma is a crude version of a theorem by Z.-X. Wen and Z.-Y. Wen [16].

Lemma 12. *Let $w \in F$ whose $a^{m_1}b^{n_1}a^{m_2}b^{n_2}\ldots a^{m_k}b^{n_k}$ one of its cyclic reductions. Then one has*

$$P_w(x,y,z) = \pm x^\mu y^\nu z^k + \sum_{\substack{0 \leqslant i \leqslant \mu,\ 0 \leqslant j \leqslant \nu \\ i+j < \mu+\nu}} \alpha_{i,j} x^i y^j z^k + \sum_{0 \leqslant j < k} A_j(x,y) z^j, \tag{10}$$

where $A_j \in \mathbb{Z}[x,y]$, $\mu = \sum_{j=1}^k |m_j| - k$, and $\nu = \sum_{j=1}^k |n_j| - k$.

If $w = a^{m_1}$, formula (10) *holds with $k = n_1 = 0$, $\mu = |m_1|$, and the like for $w = b^{m_1}$.*

As a consequence, for any $u, w \in F$, $P_u = \pm P_w$ implies $\|u\| = \|w\|$, where $\|u\|$ is the length of any cyclic reduction of u.

Lemma 13. *Let $\sigma \in \operatorname{End} F$. If M_σ is invertible and Q_σ is not identically 0, then $\sigma P = 0$ implies $P = 0$ (where $P \in \mathbb{C}[x, y, z]$).*

Proof. Suppose that $\sigma P = 0$. Then $P \circ \Pi \circ {}^{\mathrm{t}}M_\sigma = 0$. But ${}^{\mathrm{t}}M_\sigma$ is invertible, so $P \circ \Pi = 0$, which implies $P = \lambda P_1$. As $\sigma\lambda = \lambda Q_\sigma \neq 0$, one has $\sigma P_1 = 0$. By repeating this argument, one gets $P = c\lambda^n$. This proves that $P = 0$. □

Lemma 14. *Let $\sigma \in \operatorname{End} F$. If M_σ is invertible and Q_σ is not identically 0 or 1, then $\sigma^n Q_\sigma$ is not constant for any $n \geqslant 0$.*

Proof. Since σ is one-to-one if and only if σ^n $(n > 0)$ is, we see, if $\sigma\lambda \not\equiv 0$, then there does not exist $n > 0$ such that $\sigma^n\lambda \equiv 0$.

Suppose on the contrary, $Q_\sigma \not\equiv$ const and there exists $n > 1$ such that $\sigma^n Q_\sigma \equiv$ const. One has

$$\sigma^{n+1}\lambda = \sigma^n(\lambda.Q_\sigma) = (\sigma^n\lambda).(\sigma^n Q_\sigma). \tag{11}$$

Since $\lambda = P_{aba^{-1}b^{-1}} - 2$, where $P_{aba^{-1}b^{-1}}$ is the trace polynomial of the commutator $aba^{-1}b^{-1}$, one has, for any $k \geqslant 1$, $\sigma^k\lambda = P_{\sigma^k(aba^{-1}b^{-1})} - 2$.

As $\sigma^n\lambda + 2$ and $\sigma^{n+1}\lambda + 2$ are the trace polynomials of $\sigma^n(aba^{-1}b^{-1})$ and $\sigma^{n+1}(aba^{-1}b^{-1})$ respectively, formula (11) and Lemma 12 yield $\sigma^n Q_\sigma = \pm 1$. So $Q_\sigma \not\equiv$ const and $\sigma^n Q_\sigma \equiv$ const imply $\sigma^n\lambda = \pm(\sigma^{n+1}\lambda)$, which can be written as $\sigma^n(\lambda \mp \sigma\lambda) = 0$ or $\sigma^n(1 \mp Q_\sigma) = 0$. Due to Lemma 13, this means $Q_\sigma = \pm 1$. The hypothesis excludes $Q_\sigma = 1$, and $Q_\sigma = -1$ cannot happen (due to Theorem 2(2)). □

Lemma 15. *Let $\sigma \in \operatorname{End} F$. If σ is Nielsen reduced and Q_σ is not identically 0 nor 1, then $\sigma^n Q_\sigma$ is not constant for any $n \geqslant 0$.*

Proof. The proof begins as the one of Lemma 14: we have $\sigma^n\lambda = \pm(\sigma^{n+1}\lambda)$.

To say that σ is Nielsen reduced means, if we set $v_1 = \sigma(a)$, $v_2 = \sigma(b)$ and $U = \{v_1, v_2, v_1^{-1}, v_2^{-1}\}$, that, for all triples $s_1, s_2, s_3 \in U$, the following conditions hold

(N0) $s_1 \neq \varepsilon$;
(N1) $s_1s_2 \neq \varepsilon$ implies $|s_1s_2| > |s_1|, |s_2|$;
(N2) $s_1s_2 \neq \varepsilon$ and $s_2s_3 \neq \varepsilon$ implies $|s_1s_2s_3| > |s_1| - |s_2| + |s_3|$.

By the remark following Lemma 12,

$$\left\|\sigma^n(aba^{-1}b^{-1})\right\| = \left\|\sigma^{n+1}(aba^{-1}b^{-1})\right\|.$$

Let $u = \sigma^n(aba^{-1}b^{-1})$. We have

$$0 < \|u\| \leqslant \left\|\sigma(u)\right\| = \|u\|,$$

where the second inequality is due to (N2).

For any $s \in U$, we define $\xi(s)$ to be the longest prefix of s that can be cancelled in any word of the form vs, where $v \in U$ and $vs \neq \varepsilon$.

By (N2), we can denote by $v_1 = u_1\alpha u_2^{-1}$ and $v_2 = u_3\beta u_4^{-1}$, where $u_1 = \xi(v_1)$, $u_2 = \xi(v_1^{-1})$, $u_3 = \xi(v_2)$, $u_4 = \xi(v_2^{-1})$. And also by (N2), $|\alpha| \geqslant 1$ and $|\beta| \geqslant 1$.

For any $w \in F$, we define $L(w)$ to be a vector in $\mathbb{Z}^2$, with the entry of index $i \in \mathcal{A}$ is the sum of the exponents of the letter i in the word w. One has, for any $\eta \in \operatorname{End} F$, $L(\eta w) = M_\eta \cdot L(w)$. One sees $L(aba^{-1}b^{-1}) = L(u) = {}^{\mathrm{t}}(0,0)$.

So $\|u\| > 0$ implies there are both $a^{\pm 1}$ and $b^{\pm 1}$ in cyclic reduced form of u. $\|u\| = \|\sigma(u)\|$ implies that $|\alpha| = |\beta| = 1$. Moreover, $u_1 = u_3$ or $u_1 = u_4$, $u_2 = u_4$ or $u_2 = u_3$.

Suppose $u_1 = u_2 = u_3 = u_4$. If $\alpha = \beta^{\pm 1}$ then $\|\sigma(u)\| = 0$, which is a contradiction. If $\alpha \neq \beta^{\pm 1}$ then σ is invertible, which is a contradiction.

So without loosing generality, we can suppose $u_1 = u_3 \neq u_2 = u_4$. $\|u\| = \|\sigma(u)\|$ implies $u = w^{-1}(ab^{-1})^m w$ for some $w \in F$ and some $m \geqslant 1$. But notice that

$$L\big(w^{-1}(ab^{-1})^m w\big) = {}^{\mathrm{t}}(m, -m).$$

It is also a contradiction. So if $Q_\sigma \not\equiv \text{const}$, then there does not exist $n > 1$ such that $\sigma^n Q_\sigma \equiv \text{const}$. This proves the lemma. □

Proof of Theorem 3. Let P be a non-constant periodic polynomial. Due to Lemma 7, $P = c + \lambda P_1$; So, for any n, $\sigma^n\lambda$ divides $P - c$. But, as

$$\sigma^n\lambda = \lambda \cdot \prod_{k=0}^{n-1}\big(\sigma^k Q_\sigma\big),$$

Lemmas 14 and 15 implies that the degree of $\sigma^n\lambda$ is not bounded, which leads to a contradiction. □

Proposition 16. *Let $\sigma \in \operatorname{End} F$ with M_σ invertible and $Q_\sigma \not\equiv 0$ or 1. Suppose that 1 is an eigenvalue of M_{σ^2} and that the corresponding eigenspace has dimension 1. Let $\tau \in \operatorname{Aut} F$ satisfy $M_{\tau^{-1}\sigma^2\tau} = \left[\begin{smallmatrix} 1 & l \\ 0 & k \end{smallmatrix}\right]$. If $\tau^{-1}x$ is σ^2-invariant, then $\mathcal{R}_\sigma = \mathbb{C}[\tau^{-1}x]$.*

Proof. The proof follows the same line as the one of the third assertion of Theorem 2. □

Proposition 17. *Let $\sigma \in \operatorname{End} F$ with $Q_\sigma \not\equiv 0$ or 1. Suppose both eigenvalues of M_σ are roots of 1 and M_σ is similar to a diagonal matrix. Let $k \geqslant 1$ satisfy $M_\sigma^k = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. If $\sigma^k x = x$ and $\sigma^k y = y$, then $\mathcal{R}_\sigma = \mathbb{C}[x, y]$.*

Proof. Let $\sigma_1 = \sigma^k$. Then $Q_\sigma \not\equiv \text{const}$ implies $Q_{\sigma_1} \not\equiv \text{const}$. Hence (due to Lemma 14) $\deg \sigma_1\lambda > \deg\lambda$; Since x, y are σ_1-invariant, one has $\deg \sigma_1 z > 1$.

If p is a σ_1^n-invariant polynomial, then $\sigma_1^n p(x,y,z) = p(x,y,\sigma_1^n z)$. So, if $\deg_z p \geqslant 1$, then $\deg_z \sigma_1^n p > \deg_z p$ if $n > 0$. This means that any periodic polynomial is independent of z. □

Remark. There still are some cases not elucidated:

- σ and M_σ are non-invertible and σ is not Nielsen reduced;
- $\sigma = (a^2ba^{-1}b^{-1}, b^2)$, which is the case discussed in Proposition 16, except that $\sigma x \neq x$;
- $\tau = (a^2ba^{-1}b^{-1}, a^{-1}b^{-1}ab^2)$ or $\tau = (a, ab^2a^{-1}b^{-1})$, which is the case discussed in Proposition 17, except that $\tau x \neq x$ or $\tau y \neq y$.

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