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## HANKEL DETERMINANTS OF THE THUE-MORSE SEQUENCE

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### 0. Introduction.

Let  $S = \{a, b\}$  be a two-letter alphabet and  $S^*$  the free monoid generated by  $S$ . Consider the endomorphism  $\theta$  defined on  $S^*$  by

$$\theta: a \mapsto ab, b \mapsto ba.$$

Since the word  $\theta^n(a)$  is the left half of the word  $\theta^{n+1}(a)$ , it has a limit as  $n$  goes to infinity: the infinite sequence  $\epsilon = \epsilon_0\epsilon_1 \cdots \epsilon_n \cdots \in \{a, b\}^{\mathbb{N}}$  which is called the Thue-Morse (or sometimes the Prouhet-Thue-Morse) sequence.

In this article, except in Section 4, we take  $a = 1$ ,  $b = 0$ . Then the sequence  $\epsilon$  satisfies the following relations:  $\epsilon_0 = 1$ ,  $\epsilon_{2n} = \epsilon_n$ ,  $\epsilon_{2n+1} = 1 - \epsilon_n$ .

The study of the Thue-Morse sequence has been initiated by Thue (1906, [14]; 1912, [15]), who proved that it does not contain three consecutive identical blocks. A few years later, Morse (1921, [10]) studied the topological dynamical system generated by this sequence, and Gottschalk (1963, [9]) studied this sequence in the framework of minimal sets. In the last ten years, it occurred in many different fields of mathematics — ergodic theory, finite automata theory, formal language theory, number theory, algebraic

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formal power series over  $\text{GF}_2(X)$  — and also in physics in relation to quasicrystals (see, for instance, [1], [7], [8], [11], [16], [17], [18]).

In this article, we discuss some new properties of the Thue-Morse sequence.

Let  $u = (u_k)_{k \geq 0}$  be a sequence of complex numbers; then the  $(p, n)$ -order Hankel matrix associated with the sequence  $u$  is defined to be

$$H_n^p = \begin{pmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{pmatrix},$$

where  $n \geq 1$  and  $p \geq 0$ . The determinant of this matrix, denoted by  $|H_n^p|$ , is called the  $(p, n)$ -order Hankel determinant of the sequence  $u$ . The properties of Hankel determinants associated with a sequence are closely connected to the study of the moment problem, to Padé approximation, and to combinatorial properties of the sequence.

Here we consider  $\mathcal{E}_n^p$ , the  $(p, n)$ -order Hankel matrix of the Thue-Morse sequence. We denote by  $|\mathcal{E}_n^p|$  its  $(p, n)$ -order Hankel determinant. Our purpose is to study the properties of the double sequence  $(|\mathcal{E}_n^p|)_{n \geq 1, p \geq 0}$ . Figure 1 on next page shows  $|\mathcal{E}_n^p|$  modulo 2 (0's are replaced by a dot, 1's by nothing) for  $1 \leq n \leq 96$  and  $0 \leq p \leq 127$ .

This article is organized as follows. Definitions and preliminaries are given in Section 1. Section 2 is mainly devoted to establishing recurrence formulae for the sequence modulo 2 of Hankel determinants associated with the Thue-Morse sequence. Automaticity properties of the sequence of these determinants modulo 2 are established in Section 3. Further properties and applications (non-repetition in the Thue-Morse sequence and existence of some Padé approximants) are given in Section 4.

## 1. Preliminaries.

Let  $\epsilon = \epsilon_0 \epsilon_1 \cdots \epsilon_n \cdots \in \{0, 1\}^{\mathbb{N}}$  be the Thue-Morse sequence, defined by the following recurrence equations:

$$(1) \quad \epsilon_0 = 1, \quad \epsilon_{2n} = \epsilon_n, \quad \epsilon_{2n+1} = 1 - \epsilon_n, \quad \text{for } n \geq 0.$$

As we shall see below, in order to determine the Hankel determinants associated with  $\epsilon$ , we have to calculate simultaneously those associated with another sequence  $\delta = \delta_0 \delta_1 \cdots \delta_n \cdots$  which is defined by  $\delta_n = \epsilon_{n+1} - \epsilon_n$ .

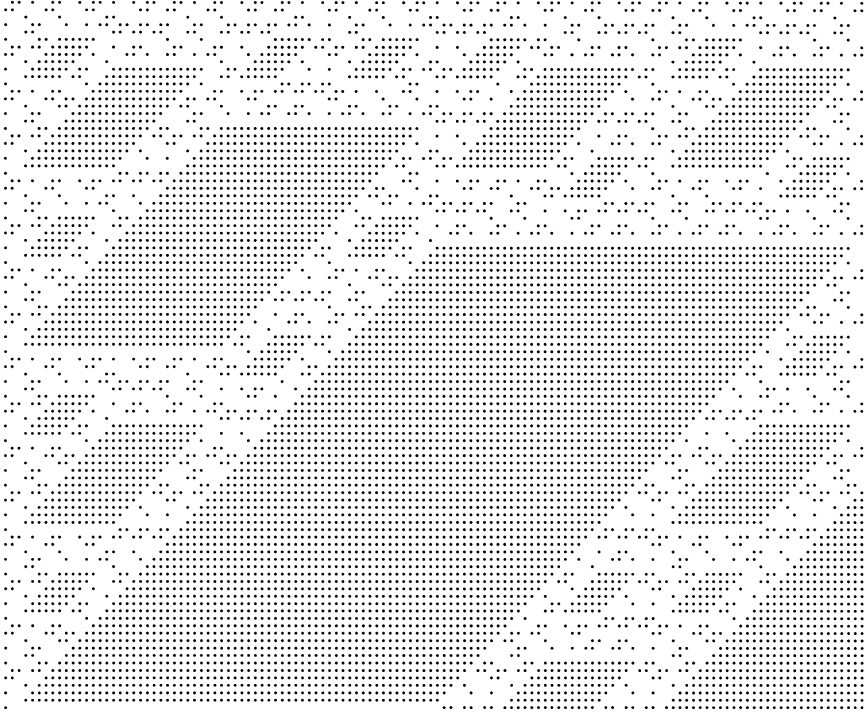


Figure 1. The set  $|\mathcal{E}_n^p|$  modulo 2 for  $1 \leq n \leq 96$  and  $0 \leq p \leq 127$ .

By (1) and the definition of  $\delta$ , we have, for  $n \geq 1$ ,

$$(2) \quad \begin{cases} \delta_{2n} + \delta_{2n+1} = \delta_n, \\ \delta_{2n} = 1 - 2\epsilon_n. \end{cases}$$

*Remark 1.1.* — The Thue-Morse sequence can be generated by the endomorphism of  $\{0, 1\}^*$  defined by  $1 \mapsto 10, 0 \mapsto 01$ . The above sequence  $\delta$  reduced modulo 2 is called the period-doubling sequence (some authors also call it the Toeplitz sequence). Like the Thue-Morse sequence, it can be generated by an endomorphism of  $\{0, 1\}^*$ :  $1 \mapsto 10, 0 \mapsto 11$ . Furthermore, it can be “induced” from the Thue-Morse sequence in the following way. Define the map  $\Phi: \{0, 1\}^2 \rightarrow \{0, 1\}$  by  $\Phi(00) = \Phi(11) = 0, \Phi(01) = \Phi(10) = 1$ . Let  $(\Psi_k)_{k \geq 0}$  be the sequence defined by  $\Psi_k = \Phi(\epsilon_k \epsilon_{k+1})$ , where  $\epsilon_k \epsilon_{k+1}$  runs through the blocks of two consecutive letters occurring in the Thue-Morse sequence. Then the sequence  $(\Psi_k)_{k \geq 0}$  is nothing but the period-doubling sequence.

*Notations.*

Throughout this paper, we adopt the following definitions and notations:

- The Thue-Morse sequence, the period-doubling sequence, and their corresponding  $(p, n)$ -order Hankel matrices (where  $p \geq 0$ ,  $n \geq 1$ ) are denoted respectively by  $\epsilon$ ,  $\delta$ ,  $\mathcal{E}_n^p$ , and  $\Delta_n^p$ .

- For a square matrix  $A$ , let  $|A|$  and  $A^t$  stand respectively for its determinant and the transposed matrix.

- $\mathbf{1}_{m,n}$  (resp.  $\mathbb{O}_{m,n}$ ) is the  $m \times n$  matrix with all its entries equal to 1 (resp. 0).

- If  $A$  is a square matrix of order  $n$ ,  $\bar{A}$  stands for the matrix

$$\begin{pmatrix} A & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & 0 \end{pmatrix},$$

and  $A^{(j)}$  for the  $n \times (n - 1)$  matrix obtained by deleting the  $j$ -th column of  $A$ .

- The symbol  $\equiv$ , unless otherwise stated, means equality modulo 2 throughout this article.

- $P_1(n) = (e_1, e_3, \dots, e_{2\lfloor \frac{n+1}{2} \rfloor - 1}, e_2, e_4, \dots, e_{2\lfloor \frac{n}{2} \rfloor})$ , where  $e_j$  is the  $j$ -th unit column vector of order  $n$ , i.e., the column vector with 1 as its  $j$ -th entry and zeros elsewhere. If no confusion can occur, we simply write  $P_1$ .

- $P_2(2n) = \begin{pmatrix} I_n & I_n \\ \mathbb{O}_{n,n} & I_n \end{pmatrix}$ , where  $I_n$  denotes the  $n \times n$  unit matrix.

- $P_2(2n + 1) = \begin{pmatrix} I_n & \mathbb{O}_{n,1} & I_n \\ \mathbb{O}_{1,n} & 1 & \mathbb{O}_{1,n} \\ \mathbb{O}_{n,n} & \mathbb{O}_{n,1} & I_n \end{pmatrix}$ .

- $P(n) = P_2(n)P_1(n)$ .

We clearly have

$$(3) \quad |P_1| = |P_2| = |P| = \pm 1.$$

*Some lemmas on matrices.*

LEMMA 1.1. — *Let  $A$  and  $B$  be two square matrices of respective orders  $m$  and  $n$ ,  $X$  a  $1 \times m$  matrix, and  $Y$  a  $1 \times n$  matrix. Then, we have*

$$\left| \begin{array}{cc} A & Y \otimes \mathbf{1}_{m,1} \\ X \otimes \mathbf{1}_{n,1} & B \end{array} \right| = |A| \cdot |B| - \left| \begin{array}{cc} A & \mathbf{1}_{m,1} \\ X & 0 \end{array} \right| \cdot \left| \begin{array}{cc} 0 & Y \\ \mathbf{1}_{n,1} & B \end{array} \right|.$$

*Proof.* — Set  $D = \begin{vmatrix} A & Y \otimes \mathbf{1}_{m,1} \\ X \otimes \mathbf{1}_{n,1} & B \end{vmatrix}$ . As a function of  $A$  and  $X$ , this is a multilinear alternating form of the columns of the matrix  $\begin{pmatrix} A \\ X \end{pmatrix}$ . Therefore it is of the form  $\begin{vmatrix} A & V \\ X & \alpha \end{vmatrix}$ , where  $V$  is a  $m \times 1$  matrix. But, since permuting two rows of  $A$  only changes the sign of  $D$ , it follows that  $V = \beta \mathbf{1}_{m,1}$ . Therefore we have

$$D = \alpha |A| + \beta \begin{vmatrix} A & \mathbf{1}_{m,1} \\ X & 0 \end{vmatrix}.$$

By taking  $X = 0$ , we get  $\alpha = |B|$ . To show the equality  $\beta = - \begin{vmatrix} 0 & Y \\ \mathbf{1}_{n,1} & B \end{vmatrix}$ , it suffices to take  $A = \begin{pmatrix} \mathbb{O}_{m-1,1} & I_{m-1} \\ 0 & \mathbb{O}_{1,m-1} \end{pmatrix}$  and  $X = (1 \ 0 \ 0 \ \dots)$ .  $\square$

As a corollary we have the following lemma.

LEMMA 1.2. — *Let  $A$  and  $B$  be two square matrices of order  $m$  and  $n$  respectively, and  $a, b, x$  and  $y$  four numbers. One has*

$$\begin{vmatrix} aA & y\mathbf{1}_{m,n} \\ x\mathbf{1}_{n,m} & bB \end{vmatrix} = a^m b^n |A| \cdot |B| - xy a^{m-1} b^{m-1} |\bar{A}| \cdot |\bar{B}|.$$

LEMMA 1.3. — *Let  $A, B$ , and  $C$  be three square matrices of order  $m, n$ , and  $p$  respectively, and three numbers  $a, b$ , and  $c$ . One has*

$$\begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = |A| \cdot |B| \cdot |C| - a^2 |A| \cdot |\bar{B}| \cdot |\bar{C}| - b^2 |\bar{A}| \cdot |B| \cdot |\bar{C}| \\ - c^2 |\bar{A}| \cdot |\bar{B}| \cdot |C| - 2abc |\bar{A}| \cdot |\bar{B}| \cdot |\bar{C}|.$$

*Proof.* — We have, provided that  $b \neq 0$ ,

$$\begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = b^{2p} c^{-2p} \begin{vmatrix} A & c\mathbf{1}_{m,n} & c\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & acb^{-1}\mathbf{1}_{n,p} \\ c\mathbf{1}_{m,p} & acb^{-1}\mathbf{1}_{n,p} & c^2 b^{-2} C \end{vmatrix} \\ = b^{2p} c^{-2p} |A| \cdot \begin{vmatrix} B & acb^{-1}\mathbf{1}_{n,p} \\ acb^{-1}\mathbf{1}_{n,p} & c^2 b^{-2} C \end{vmatrix} \\ - b^{2p} c^{2-2p} |\bar{A}| \cdot \begin{vmatrix} B & acb^{-1}\mathbf{1}_{n,p} & \mathbf{1}_{n,1} \\ acb^{-1}\mathbf{1}_{n,p} & c^2 b^{-2} C & \mathbf{1}_{p,1} \\ \mathbf{1}_{1,n} & \mathbf{1}_{1,p} & 0 \end{vmatrix},$$

(the second equality results from Lemma 1.2), from which we deduce the formula

$$(4) \quad \begin{vmatrix} A & c\mathbf{1}_{m,n} & b\mathbf{1}_{m,p} \\ c\mathbf{1}_{m,n} & B & a\mathbf{1}_{n,p} \\ b\mathbf{1}_{m,p} & a\mathbf{1}_{n,p} & C \end{vmatrix} = |A| \cdot \begin{vmatrix} B & a\mathbf{1}_{n,p} \\ a\mathbf{1}_{n,p} & C \end{vmatrix} - |\bar{A}| \cdot \begin{vmatrix} B & a\mathbf{1}_{n,p} & c\mathbf{1}_{n,1} \\ a\mathbf{1}_{n,p} & C & b\mathbf{1}_{p,1} \\ c\mathbf{1}_{n,1} & b\mathbf{1}_{p,1} & 0 \end{vmatrix}$$

which is valid without any restriction on  $b$ .

The last determinant of Formula (4) above can be itself computed by using (4) two more times.  $\square$

Lemma 1.3 can be extended in the following way. Although we shall not use this extension, we mention it because it could be of interest in similar situations. Let  $\{a_{i,j}\}_{1 \leq i,j \leq n}$  be a collection of numbers such that  $a_{i,i} = 0$ , and  $\{A_i\}_{1 \leq i \leq n}$  a sequence of square matrices of respective dimensions  $m_i$ .

Define a matrix  $M$  by blocks: put the  $A$ 's on the diagonal, and the block  $a_{i,j}\mathbf{1}_{m_i,m_j}$  at position  $(i,j)$  for  $i \neq j$ . Then

$$|M| = \sum_{\sigma} \varepsilon_{\sigma} \left[ \prod_{i=\sigma(i)} |A_i| \right] \prod_{i \neq \sigma(i)} (-|\bar{A}_i| a_{i,\sigma(i)}).$$

LEMMA 1.4. — *Let  $x \in \mathbb{R}$ , and let  $A$  be an  $m \times m$  matrix, then*

- (i)  $|x\mathbf{1}_{m,m} + A| = |A| - x|\bar{A}|$ ,
- (ii)  $|\overline{x\mathbf{1}_{m,m} + A}| = |\bar{A}|$ ,
- (iii)  $|\overline{-A}| = (-1)^{m+1}|\bar{A}|$ .

*Proof.* — To prove (i), write

$$|x\mathbf{1}_{m,m} + A| = \begin{vmatrix} x\mathbf{1}_{m,m} + A & \mathbb{O}_{m,1} \\ \mathbf{1}_{1,m} & 1 \end{vmatrix} = \begin{vmatrix} A & -x\mathbf{1}_{m,1} \\ \mathbf{1}_{1,m} & 1 \end{vmatrix},$$

and conclude by using Lemma 1.2.

Assertions (ii) and (iii) are obvious.  $\square$

*Remark 1.2.* — Because they express identities between polynomials with integer coefficients, the preceding lemmas are valid for matrices with entries in any commutative ring.

## 2. Fundamental recurrence equations.

The aim of this section is to determine recurrence formulae for the sequence  $|\mathcal{E}_n^p|$  ( $n \geq 1, p \geq 0$ ), which will play an essential rôle in this paper. We find that, in order to establish such formulae, we need to distinguish different cases according to the parities of  $n$  and  $p$ . These formulae involve the quantities  $|\overline{\mathcal{E}_n^p}|$ ,  $|\Delta_n^p|$  and  $|\overline{\Delta_n^p}|$ . Hence, we shall simultaneously establish recurrence formulae for all these sixteen quantities, thus obtaining the fundamental results of this section.

THEOREM 2.1. — *For  $p \geq 0$  and  $n \geq 1$ , one has*

- 1) 
$$\begin{aligned} |\mathcal{E}_{2n}^{2p}| &= |\mathcal{E}_n^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\mathcal{E}_n^p| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$
- 2) 
$$\begin{aligned} |\overline{\mathcal{E}_{2n}^{2p}}| &= 4|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}| + |\overline{\mathcal{E}_n^p}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\overline{\mathcal{E}_n^p}| \cdot |\Delta_n^p|, \end{aligned}$$
- 3) 
$$\begin{aligned} |\mathcal{E}_{2n+1}^{2p}| &= |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$
- 4) 
$$\begin{aligned} |\overline{\mathcal{E}_{2n+1}^{2p}}| &= 4|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^p}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^p}| \\ &\equiv |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^p|, \end{aligned}$$
- 5) 
$$\begin{aligned} |\mathcal{E}_{2n}^{2p+1}| &= (-1)^n \{ |\mathcal{E}_n^{p+1}| \cdot |\Delta_n^p| - 2|\mathcal{E}_n^{p+1}| \cdot |\overline{\Delta_n^p}| \\ &\quad + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}| \} \\ &\equiv |\mathcal{E}_n^{p+1}| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| + |\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}|, \end{aligned}$$
- 6) 
$$\begin{aligned} |\overline{\mathcal{E}_{2n}^{2p+1}}| &= (-1)^n \{ 4|\mathcal{E}_n^{p+1}| \cdot |\overline{\Delta_n^p}| - |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p| + 2|\overline{\mathcal{E}_n^{p+1}}| \cdot |\overline{\Delta_n^p}| \} \\ &\equiv |\overline{\mathcal{E}_n^{p+1}}| \cdot |\Delta_n^p|, \end{aligned}$$
- 7) 
$$\begin{aligned} |\mathcal{E}_{2n+1}^{2p+1}| &= (-1)^{n+1} \{ |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^{p+1}| - 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| \\ &\quad + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}| \} \\ &\equiv |\mathcal{E}_{n+1}^p| \cdot |\Delta_n^{p+1}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| + |\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}|, \end{aligned}$$
- 8) 
$$\begin{aligned} |\overline{\mathcal{E}_{2n+1}^{2p+1}}| &= (-1)^{n+1} \{ 4|\mathcal{E}_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| - |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}| \\ &\quad + 2|\overline{\mathcal{E}_{n+1}^p}| \cdot |\overline{\Delta_n^{p+1}}| \} \\ &\equiv |\overline{\mathcal{E}_{n+1}^p}| \cdot |\Delta_n^{p+1}|, \end{aligned}$$

- 9)  $|\Delta_{2n}^{2p}| = (-1)^n |\Delta_n^p|^2$   
 $\equiv |\Delta_n^p|,$
- 10)  $|\overline{\Delta_{2n}^{2p}}| \equiv 0,$
- 11)  $|\Delta_{2n+1}^{2p}| = (-1)^n \{ |\overline{\mathcal{E}_{n+1}^p}|^2 + 2|\mathcal{E}_{n+1}^p| \cdot |\overline{\mathcal{E}_{n+1}^p}| \}$   
 $\equiv |\overline{\mathcal{E}_{n+1}^p}|,$
- 12)  $|\overline{\Delta_{2n+1}^{2p}}| \equiv |\overline{\mathcal{E}_{n+1}^p}|,$
- 13)  $|\Delta_{2n}^{2p+1}| \equiv |\Delta_n^p| \cdot |\Delta_n^{p+1}| + |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$
- 14)  $|\overline{\Delta_{2n}^{2p+1}}| \equiv |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$
- 15)  $|\Delta_{2n+1}^{2p+1}| \equiv |\Delta_{n+1}^p| \cdot |\Delta_{n+1}^{p+1}| + |\Delta_{n+1}^p| \cdot |\overline{\Delta_{n+1}^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_{n+1}^{p+1}|,$
- 16)  $|\overline{\Delta_{2n+1}^{2p+1}}| \equiv |\Delta_{n+1}^p| \cdot |\overline{\Delta_{n+1}^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_{n+1}^{p+1}|.$

*Proof.* — First we are going to establish a few general properties of Hankel matrices associated with a sequence  $\{u_j\}_{j \geq 0}$ . For  $n \geq 1$  and  $p \geq 0$  we consider the Hankel matrix  $H_n^p = (u_{p+i+j-2})_{1 \leq i, j \leq n}$ , together with the matrix  $K_n^p = (u_{p+2(i+j-2)})_{1 \leq i, j \leq n}$ .

When  $u = \varepsilon$  is the Thue-Morse sequence, one has

$$(5) \quad K_n^{2p} = \mathcal{E}_n^p \quad \text{and} \quad K_n^{2p+1} = \mathbf{1}_{n,n} - \mathcal{E}_n^p.$$

When  $u_n = \delta_n (= \varepsilon_{n+1} - \varepsilon_n)$ , one has

$$(6) \quad K_n^{2p} = \mathbf{1}_{n,n} - 2\mathcal{E}_n^p \quad \text{and} \quad K_n^{2p+1} = \Delta_n^p + 2\mathcal{E}_n^p - \mathbf{1}_{n,n}.$$

Let  $M = (m_{i,j})_{1 \leq i, j \leq n}$  be any  $n \times n$ -matrix. Let  $\nu = [\frac{1}{2}(n+1)]$  and  $\mu = [\frac{1}{2}n]$ . One can easily check the following formula

$$(7) \quad P_1^t M P_1 = \begin{pmatrix} (m_{2i-1, 2j-1})_{\substack{1 \leq i \leq \nu \\ 1 \leq j \leq \nu}} & (m_{2i-1, 2j})_{\substack{1 \leq i \leq \nu \\ 1 \leq j \leq \mu}} \\ (m_{2i, 2j-1})_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \nu}} & (m_{2i, 2j})_{\substack{1 \leq i \leq \mu \\ 1 \leq j \leq \mu}} \end{pmatrix}$$

from which one can get

$$(8) \quad P_1^t H_{2n}^p P_1 = \begin{pmatrix} K_n^p & K_n^{p+1} \\ K_n^{p+1} & K_n^{p+2} \end{pmatrix}$$

and

$$(9) \quad \begin{aligned} P_1^t H_{2n+1}^p P_1 &= \begin{pmatrix} K_{n+1}^p & (K_{n+1}^{p+1})^{(n+1)} \\ (K_{n+1}^{p+1})^{(n+1)t} & K_n^{p+2} \end{pmatrix} \\ &= \begin{pmatrix} K_{n+1}^p & (K_{n+1}^p)^{(1)} \\ (K_{n+1}^p)^{(1)t} & K_n^{p+2} \end{pmatrix}. \end{aligned}$$

In other words,  $P_1^t H_{2n+1}^p P_1$  is obtained by removing the last row and the last column from the matrix  $P_1^t H_{2(n+1)} P_1$ .

*Proof of 1).* — By using (8) and (5), we have

$$P_1^t \mathcal{E}_{2n}^{2p} P_1 = \begin{pmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} - \mathcal{E}_n^p \\ \mathbf{1}_{n,n} - \mathcal{E}_n^p & \mathcal{E}_n^{p+1} \end{pmatrix}.$$

Then by the definition of  $P_2$  and  $\Delta_n^p$ , we have

$$(10) \quad P_2^t P_1^t \mathcal{E}_{2n}^{2p} P_1 P_2 = \begin{pmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{pmatrix}.$$

Hence by (3) and Lemmas 1.2 and 1.4, we obtain

$$\begin{aligned} |\mathcal{E}_{2n}^{2p}| &= \left| \begin{array}{cc} \mathcal{E}_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{array} \right| \\ &= |\mathcal{E}_n^p| \cdot |2\mathbf{1}_{n,n} + \Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{2\mathbf{1}_{n,n} + \Delta_n^p}| \\ &= |\mathcal{E}_n^p| (|\Delta_n^p| - 2|\overline{\Delta_n^p}|) - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| \\ &= |\mathcal{E}_n^p| \cdot |\Delta_n^p| - |\overline{\mathcal{E}_n^p}| \cdot |\overline{\Delta_n^p}| - 2|\mathcal{E}_n^p| \cdot |\overline{\Delta_n^p}|. \end{aligned}$$

The proofs of the other assertions follow the same lines: to compute the determinant of a matrix  $A$ , we find a matrix  $Q$  such that the determinant of  $Q A Q^t$  is computable, for instance by using Lemmas 1.2, 1.3, and 1.4.

In the sequel, we only write the transformation matrices and omit the details of the calculations.

*Proof of 2).* — One has

$$\begin{pmatrix} P^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n}^{2p} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} = \begin{pmatrix} P^t \mathcal{E}_{2n}^{2p} P & P^t \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} P & 0 \end{pmatrix}.$$

Therefore, taking into account (10),

$$|\bar{D}| = \begin{vmatrix} \mathcal{E}_n^p & \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & 2\mathbf{1}_{n,n} + \Delta_n^p & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & 2\mathbf{1}_{1,n} & 0 \end{vmatrix}.$$

*Proof of 3).* — Computing as in 1), we have successively

$$P_1^t \mathcal{E}_{2n+1}^{2p} P_1 = \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} - (\mathcal{E}_{n+1}^p)^{(n+1)} \\ (\mathbf{1}_{n+1,n} - (\mathcal{E}_{n+1}^p)^{(n+1)})^t & \mathcal{E}_n^{p+1} \end{pmatrix},$$

and

$$(11) \quad P_2^t P_1^t \mathcal{E}_{2n+1}^{2p} P_1 P_2 = \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^p \end{pmatrix}.$$

*Proof of 4).* — One has, due to (11),

$$\begin{aligned} & \begin{pmatrix} P^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n+1}^{2p} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \\ & = \begin{pmatrix} \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^p & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & 2\mathbf{1}_{1,n} & 0 \end{pmatrix}. \end{aligned}$$

*Proof of 5).* — One has

$$(12) \quad \begin{aligned} P_2 P_1^t \mathcal{E}_{2n}^{2p+1} P_1 P_2^t &= P_2 \begin{pmatrix} \mathbf{1}_{n,n} - \mathcal{E}_n^p & \mathcal{E}_n^{p+1} \\ \mathcal{E}_n^{p+1} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix} P_2^t \\ &= \begin{pmatrix} 2\mathbf{1}_{n,n} + \Delta_n^p & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix}. \end{aligned}$$

*Proof of 6).* — Due to (12), one has

$$\begin{aligned} & \begin{pmatrix} P_2 P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n}^{2p+1} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 P_2^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \\ & = \begin{pmatrix} 2\mathbf{1}_{n,n} + \Delta_n^p & \mathbf{1}_{n,n} & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} & \mathbf{1}_{n,1} \\ 2\mathbf{1}_{1,n} & \mathbf{1}_{1,n} & 0 \end{pmatrix}. \end{aligned}$$

*Proof of 7).* — Let

$$Q_1 = \begin{pmatrix} 1 & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & I_n \ I_n \\ \mathbb{O}_{n,n} & I_n \end{pmatrix},$$

then

$$(13) \quad Q_1^t P_1^t \mathcal{E}_{2n+1}^{2p+1} P_1 Q_1 = Q_1^t \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & (\mathcal{E}_{n+1}^{p+1})^{(1)} \\ ((\mathcal{E}_{n+1}^{p+1})^{(1)})^t & \mathbf{1}_{n,n} - \mathcal{E}_n^{p+1} \end{pmatrix} Q_1 \\ = \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^{p+1} \end{pmatrix},$$

and Formula 2.1.7 follows as above.

*Proof of 8).* — Due to (13), one has

$$\begin{pmatrix} Q_1^t P_1^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{2n+1}^{2p+1} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P_1 Q_1 & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \\ = \begin{pmatrix} \mathbf{1}_{n+1,n+1} - \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & 2\mathbf{1}_{n,n} + \Delta_n^{p+1} & 2\mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & 2\mathbf{1}_{1,n} & 0 \end{pmatrix}.$$

*Proof of 9).* — Write

$$(14) \quad P_2^t P_1^t \Delta_{2n}^{2p} P_1 P_2 = P_2^t \begin{pmatrix} \mathbf{1}_{n,n} - 2\mathcal{E}_n^p & \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p \\ \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \end{pmatrix} P_2 \\ = \begin{pmatrix} \mathbf{1}_{n,n} - 2\mathcal{E}_n^p & \Delta_n^p \\ \Delta_n^p & \mathbb{O}_{n,n} \end{pmatrix},$$

hence

$$|\Delta_{2n}^{2p}| = (-1)^n |\Delta_n^p|^2.$$

*Proof of 10).* — Due to (14)

$$\begin{pmatrix} P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n}^{2p} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{1}_{n,n} & \Delta_n^p & \mathbf{1}_{n,1} \\ \Delta_n^p & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbf{1}_{1,n} & \mathbb{O}_{1,n} & 0 \end{pmatrix},$$

hence  $|\overline{\Delta_{2n}^{2p}}| \equiv 0$ .

*Proof of 11).* — We write

$$\begin{aligned} P_2^t P_1^t \Delta_{2n+1}^{2p} P_1 P_2 &= P_2^t \begin{pmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & B^{(n+1)} \\ (B^{(n+1)})^t & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \end{pmatrix} P_2 \\ &= \begin{pmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D \\ D^t & \mathbb{O}_{n,n} \end{pmatrix}, \end{aligned}$$

where  $B = \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} + 2\mathcal{E}_{n+1}^p$  and  $D = (\Delta_{n+1}^p)^{(n+1)}$ , hence

$$\begin{aligned} |\Delta_{2n+1}^{2p}| &= \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D \\ D^t & \mathbb{O}_{n,n} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & D & -v \\ D^t & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbb{O}_{1,n+1} & \mathbb{O}_{1,n} & 1 \end{vmatrix}, \end{aligned}$$

where  $v = (\epsilon_{p+n}, \epsilon_{p+n+1}, \dots, \epsilon_{p+2n})^t$ . Now, we add the  $(2n+2)$ -th column of the above matrix to the  $(2n+1)$ -th column, we then add the resulting  $(2n+1)$ -th column to the  $(2n)$ -th column, and we continue this procedure until the  $(n+2)$ -th column. Then, from the definition of  $\Delta_n^p$  and noticing that  $v$  is precisely the last column of  $\mathcal{E}_{n+1}^p$ , we have

$$\begin{aligned} |\Delta_{2n+1}^{2p}| &= \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p \\ D^t & \mathbb{O}_{n,n+1} \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} \end{vmatrix} \\ &= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,1} \\ D^t & \mathbb{O}_{n,n+1} & \mathbb{O}_{n,1} \\ -v^t & \mathbb{O}_{1,n+1} & 1 \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix} \\ &= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}_{n+1}^p & -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,1} \\ -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix} \\ &= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} & -\mathcal{E}_{n+1}^p & -\mathbf{1}_{n+1,1} \\ -\mathcal{E}_{n+1}^p & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ -\mathbf{1}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix} \\ &= (-1)^n \begin{vmatrix} -\mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,n+1} & -\mathbf{1}_{n+1,1} \\ \mathbb{O}_{n+1,n+1} & -\mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{1,n+1} & -\mathbf{1}_{1,n+1} & 0 \end{vmatrix} \\ &= (-1)^n \begin{vmatrix} \mathcal{E}_{n+1}^p & -\mathbf{1}_{n+1,n+1} & -\mathbf{1}_{n+1,1} \\ -\mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n+1} + \mathcal{E}_{n+1}^p & \mathbf{1}_{n+1,1} \\ -\mathbf{1}_{1,n+1} & -\mathbf{1}_{1,n+1} & 0 \end{vmatrix}. \end{aligned}$$

*Proof of 12).* — We write

$$\begin{aligned} \begin{pmatrix} P^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n+1}^{2p} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} P & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \\ \equiv \begin{pmatrix} \mathbf{1}_{n+1,n+1} & D & \mathbf{1}_{n+1,1} \\ D^t & \mathbb{O}_{n,n} & \mathbb{O}_{n,1} \\ \mathbf{1}_{1,n+1} & \mathbb{O}_{1,n} & 0 \end{pmatrix}, \end{aligned}$$

where  $D$  is defined as in 11). Therefore computing as in 11), we have

$$|\overline{\Delta_{2n+1}^{2p}}| \equiv |D \mathbf{1}_{n+1,1}| \cdot \left| \begin{matrix} D^t \\ \mathbf{1}_{1,n+1} \end{matrix} \right| \equiv |\overline{\mathcal{E}_{n+1}^p}|^2 \equiv |\overline{\mathcal{E}_{n+1}^p}|.$$

*Proof of 13).* — We have

$$\begin{aligned} P_1^t \Delta_{2n}^{2p+1} P_1 &= \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} + 2\mathcal{E}_n^p & \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} \\ \mathbf{1}_{n,n} - 2\mathcal{E}_n^{p+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} + 2\mathcal{E}_n^{p+1} \end{pmatrix} \\ &\equiv \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} & \mathbf{1}_{n,n} \\ \mathbf{1}_{n,n} & \Delta_n^{p+1} - \mathbf{1}_{n,n} \end{pmatrix}. \end{aligned}$$

*Proof of 14).* — As in 13), we have

$$\begin{aligned} \begin{pmatrix} P_1^t & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n}^{2p+1} & \mathbf{1}_{2n,1} \\ \mathbf{1}_{1,2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbb{O}_{2n,1} \\ \mathbb{O}_{1,2n} & 1 \end{pmatrix} \\ \equiv \begin{pmatrix} \Delta_n^p - \mathbf{1}_{n,n} & \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{n,n} & \Delta_n^{p+1} - \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n} & \mathbf{1}_{1,n} & 0 \end{pmatrix}. \end{aligned}$$

*Proof of 15).* — As in 13), we have

$$Q_1^t \Delta_{2n+1}^{2p+1} Q_1 \equiv \begin{pmatrix} \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n} \\ \mathbf{1}_{n,n+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} \end{pmatrix}.$$

*Proof of 16).* — As in 15), we have

$$\begin{aligned} \begin{pmatrix} Q_1^t & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \begin{pmatrix} \Delta_{2n+1}^{2p+1} & \mathbf{1}_{2n+1,1} \\ \mathbf{1}_{1,2n+1} & 0 \end{pmatrix} \begin{pmatrix} Q_1 & \mathbb{O}_{2n+1,1} \\ \mathbb{O}_{1,2n+1} & 1 \end{pmatrix} \\ \equiv \begin{pmatrix} \Delta_{n+1}^p - \mathbf{1}_{n+1,n+1} & \mathbf{1}_{n+1,n} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{n,n+1} & \Delta_n^{p+1} - \mathbf{1}_{n,n} & \mathbf{1}_{n,1} \\ \mathbf{1}_{1,n+1} & \mathbf{1}_{1,n} & 0 \end{pmatrix}. \quad \square \end{aligned}$$

*Remark 2.1.* — The Hankel determinants  $(H_n^p)_{n \geq 1, p \geq 0}$  of a sequence of complex numbers satisfy the following recurrence equation (see for example [3], p. 96),

$$|H_n^p| \cdot |H_n^{p+2}| - |H_n^{p+1}|^2 = |H_{n-1}^{p+2}| \cdot |H_{n+1}^p|.$$

Now let  $\ell, j$  and  $k$  be given; if  $|H_n^\ell| = 0$  for  $j \leq n \leq j+k$ , then by the above formula,  $|H_n^p| = 0$  on the rhombus whose vertices are  $(\ell, j)$ ,  $(\ell+k, j)$ ,  $(\ell-k, j+k)$  and  $(\ell, j+k)$ . Similarly, if  $|H_{j+1}^p| = 0$  for  $\ell-1 \leq p \leq \ell+k$ , then, either  $|H_{j-1}^p| = 0$  for  $\ell+1 \leq p \leq \ell+k+1$ , or  $|H_{j+1}^p| = 0$  for  $\ell-1 \leq p \leq \ell+k-1$ . Therefore the set of zeros of the sequence  $(|H_n^p|)_{n \geq 1, p \geq 0}$  is the union of rhombi which are separated by nonzero elements. This explains the patterns shown on Figure 1.

PROPOSITION 2.1. — *Define*

$$\begin{aligned} |\mathcal{E}_0^p| &= \begin{cases} 0 & \text{if } p = 0, \\ 1 & \text{if } p \geq 1, \end{cases} & |\overline{\mathcal{E}}_0^p| &= 1 - |\mathcal{E}_0^p|, \\ |\Delta_0^p| &= 1, & |\overline{\Delta}_0^p| &= 0 \quad \text{for } p \geq 0. \end{aligned}$$

Then formulae of Theorem 2.1 hold for  $p \geq 0$  and  $n \geq 0$ .

*Proof.* — Formulae have to be checked one by one. The conclusion results from (1), (2), and the following facts:  $|\overline{\mathcal{E}}_1^p| = |\overline{\Delta}_1^p| = -1$ ,  $|\mathcal{E}_1^p| = \varepsilon_p$ , and  $|\Delta_1^p| = \delta_p$ .  $\square$

From Proposition 2.1, we see that if we can determine the quantities involved in Theorem 2.1 for  $p = 0, 1$ , then we can determine these quantities for all  $p \geq 2$  by the recurrence equations of Proposition 2.1. Propositions 2.2 and 2.3 below are devoted to this purpose.

PROPOSITION 2.2. — *With the above notations, we have*

$$|\mathcal{E}_n^0| \equiv n, \quad |\Delta_n^0| \equiv 1, \quad |\overline{\mathcal{E}}_n^0| \equiv 1, \quad |\overline{\Delta}_n^0| \equiv n.$$

*Proof.* — For  $n = 1$ , the above equalities can be checked directly. Assume that the proposition is true for  $n \leq k$ . Then, if  $n = k+1 = 2\ell$  is even, we have by 2.1.1 and the induction hypothesis,

$$|\mathcal{E}_n^0| \equiv |\mathcal{E}_\ell^0| \cdot |\Delta_\ell^0| + |\overline{\mathcal{E}}_\ell^0| \cdot |\overline{\Delta}_\ell^0| \equiv 2\ell.$$

If  $n = k + 1 = 2\ell + 1$  is odd, then, by 2.1.3, we have

$$|\mathcal{E}_n^0| \equiv |\mathcal{E}_{\ell+1}^0| \cdot |\Delta_\ell^0| + |\overline{\mathcal{E}}_{\ell+1}^0| \cdot |\overline{\Delta}_\ell^0| \equiv \ell + 1 + \ell \equiv 2\ell + 1.$$

Thus we obtain the first assertion. The other ones can be obtained by the same method.  $\square$

PROPOSITION 2.3. — For  $p = 1$ , we have the following relations :

- 1)  $|\mathcal{E}_n^1| \equiv \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod{6}, \\ 1 & \text{otherwise;} \end{cases}$
- 2)  $|\Delta_n^1| \equiv \begin{cases} 0 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{otherwise;} \end{cases}$
- 3)  $|\overline{\mathcal{E}}_n^1| \equiv \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3}, \\ 1 & \text{otherwise;} \end{cases}$
- 4)  $|\overline{\Delta}_n^1| \equiv \begin{cases} 0 & \text{if } n \equiv 0, 5 \pmod{6}, \\ 1 & \text{otherwise.} \end{cases}$

*Proof.* — Assertions 13), 15), 14) and 16) of Theorem 2.1 yield

$$(15) \quad |\Delta_{2n}^1| \equiv |\overline{\Delta}_{2n+1}^1| \equiv (n+1)|\Delta_n^1| + |\overline{\Delta}_n^1|,$$

$$(16) \quad |\Delta_{2n+1}^1| \equiv |\overline{\Delta}_{2n}^1| \equiv n|\Delta_n^1| + |\overline{\Delta}_n^1|,$$

from which one gets

$$(17) \quad |\Delta_{4n}^1| \equiv |\Delta_n^1|,$$

$$(18) \quad |\Delta_{4n+1}^1| \equiv |\Delta_{2n+1}^1|,$$

$$(19) \quad |\Delta_{4n+2}^1| \equiv |\Delta_{2n}^1|,$$

$$(20) \quad |\Delta_{4n+3}^1| \equiv |\Delta_n^1|.$$

It is easily checked that  $|\Delta_1^1| = 0$ ,  $|\Delta_2^1| = -1$ , and  $|\Delta_3^1| = 1$ . We will prove Assertion 2 by induction. Suppose that it is true for  $1 \leq n < 4k$ , and consider four cases.

(i) By (17), and since  $4k \equiv k \pmod{3}$ , we have

$$|\Delta_{4k}^1| \equiv |\Delta_k^1| \equiv \begin{cases} 0 & \text{if } 4k \equiv 1 \pmod{3}, \\ 1 & \text{otherwise.} \end{cases}$$

(ii) By (18), and since  $4k + 1 \equiv 1 \Leftrightarrow 2k + 1 \equiv 1 \pmod{3}$ , we have

$$|\Delta_{4k+1}^1| \equiv |\Delta_{2k+1}^1| \equiv \begin{cases} 0 & \text{if } 4k + 1 \equiv 1 \pmod{3}, \\ 1 & \text{otherwise.} \end{cases}$$

(iii) By (19), and since  $4k + 2 \equiv 1 \Leftrightarrow 2k \equiv 1 \pmod{3}$ , we have

$$|\Delta_{4k+2}^1| \equiv |\Delta_{2k}^1| \equiv \begin{cases} 0 & \text{if } 4k + 2 \equiv 1 \pmod{3}, \\ 1 & \text{otherwise.} \end{cases}$$

(iv) The case  $4k + 3$  is the same as (i).

This proves Assertion 2.

By using Assertion 2 and Formulae (15) and (16), one can prove Assertion 4 by induction.

By Equalities 6) and 8) in Theorem 2.1 and Proposition 2.2 we have  $|\overline{\mathcal{E}}_{2n}^1| = |\overline{\mathcal{E}}_n^1|$  and  $|\overline{\mathcal{E}}_{2n+1}^1| = |\overline{\Delta}_n^1|$ , then by the fourth congruence in Proposition 2.2 and induction, we prove 3).

Finally, by 3), 4) and Proposition 2.2,  $|\mathcal{E}_{2n}^1| \equiv |\mathcal{E}_n^1| + (n+1)|\overline{\mathcal{E}}_n^1|$  and  $|\mathcal{E}_{2n+1}^1| \equiv n|\Delta_n^1| + |\overline{\Delta}_n^1|$ , then, by Propositions 2, 3, 4 and the same discussion as above, 1) is proved by induction.  $\square$

### Generating series

Let  $(u_n)_{n \geq 0}$  be a sequence with  $u_n \in \mathbb{F}_2$ , then the formal power series

$$u(x) = \sum_{n \geq 0} u_n x^n$$

is called the generating series of the sequence  $(u_n)_{n \geq 0}$ .

A sequence  $\{u_n\}_{n \geq 0}$  is periodic of period  $s$  if and only if its generating series adds up to a rational fraction of the form  $\frac{P(x)}{1+x^s}$ , where  $P$  is a polynomial of degree less than  $s$ .

Let  $A(x) = \sum_{n \geq 0} a_n x^n$ ,  $B(x) = \sum_{n \geq 0} b_n x^n$  be two formal power series with  $a_n, b_n \in \mathbb{F}_2$ , then their Hadamard product is defined to be

$$A(x) \star B(x) = \sum_{n \geq 0} a_n b_n x^n.$$

The Hadamard product of generating series of periodic sequences is the generating series of a periodic sequence having as a period the lowest common multiple of the periods.

For  $p = 0, 1, \dots$ , define

$$(21) \quad \begin{cases} f^{(p)}(x) = \sum_{n \geq 0} |\mathcal{E}_n^p| x^n, & g^{(p)}(x) = \sum_{n \geq 0} |\Delta_n^p| x^n, \\ \overline{f^{(p)}}(x) = \sum_{n \geq 0} |\overline{\mathcal{E}_n^p}| x^n, & \overline{g^{(p)}}(x) = \sum_{n \geq 0} |\overline{\Delta_n^p}| x^n, \end{cases}$$

where coefficients are taken modulo 2, with the convention of Proposition 2.1 when  $p = 0$ .

By Propositions 2.2 and 2.3, we have

$$(22) \quad \begin{cases} f^{(0)} = \frac{x}{1+x^2}, & \overline{f^{(0)}} = \frac{1}{1+x}, \\ g^{(0)} = \frac{1}{1+x}, & \overline{g^{(0)}} = \frac{x}{1+x^2}, \\ f^{(1)} = \frac{1+x^3+x^4+x^5}{1+x^6}, & \overline{f^{(1)}} = \frac{x+x^2}{1+x^3}, \\ g^{(1)} = \frac{1+x^2}{1+x^3}, & \overline{g^{(1)}} = \frac{x+x^2+x^3+x^4}{1+x^6}. \end{cases}$$

Recurrence formulae of Proposition 2.1 and Propositions 2.2 and 2.3 make it possible to compute the above quantities recursively. As an illustration, we compute  $g^{(2)}$  and  $f^{(2)}$ :

$$\begin{aligned} g^{(2)} &= \sum_{n \geq 0} |\Delta_n^2| x^n = \sum_{n \geq 0} |\Delta_{2n}^2| x^{2n} + \sum_{n \geq 0} |\Delta_{2n+1}^2| x^{2n+1} \\ &= \sum_{n \geq 0} |\Delta_n^1| (x^2)^n + x \sum_{n \geq 0} |\overline{\mathcal{E}_{n+1}^1}| (x^2)^n \end{aligned}$$

(by Theorem 2.1, Assertions 9 and 11)

$$(23) \quad = \frac{1+x^4}{1+x^6} + \frac{x(1+x^2)}{1+x^6} = \frac{1+x}{1+x^3}$$

(by (22));

$$\begin{aligned} f^{(2)} &= \sum_{n \geq 0} |\mathcal{E}_n^2| x^n = \sum_{n \geq 0} |\mathcal{E}_{2n}^2| x^{2n} + \sum_{n \geq 0} |\mathcal{E}_{2n+1}^2| x^{2n+1} \\ &= \sum_{n \geq 0} (|\mathcal{E}_n^1| \cdot |\Delta_n^1| + |\overline{\mathcal{E}_n^1}| \cdot |\overline{\Delta_n^1}|) x^{2n} + x \sum_{n \geq 0} (|\mathcal{E}_{n+1}^1| \cdot |\Delta_n^1| + |\overline{\mathcal{E}_{n+1}^1}| \cdot |\overline{\Delta_n^1}|) x^{2n} \end{aligned}$$

(by Theorem 2.1, Assertions 5 and 7)

$$= (f^{(1)} \star g^{(1)} + \overline{f^{(1)}} \star \overline{g^{(1)}})^2 + x(\widehat{f^{(1)}} \star g^{(1)} + \overline{\widehat{f^{(1)}}} \star \overline{g^{(1)}})$$

where, by (22),

$$(24) \quad \widehat{f^{(1)}} = \sum_{n \geq 0} |\mathcal{E}_{n+1}^1| x^n = \frac{x^2 + x^3 + x^4 + x^5}{1 + x^6},$$

and

$$(25) \quad \overline{\widehat{f^{(1)}}} = \sum_{n \geq 0} |\overline{\mathcal{E}_{n+1}^1}| x^n = \frac{1 + x}{1 + x^3}.$$

By using Equalities (22), (24), and (25), we obtain

$$\begin{aligned} f^{(1)} \star g^{(1)} &= \frac{1 + x^3 + x^5}{1 + x^6}, & \overline{f^{(1)}} \star \overline{g^{(1)}} &= \frac{x + x^2 + x^4}{1 + x^6}, \\ \widehat{f^{(1)}} \star g^{(1)} &= \frac{x^2 + x^3 + x^5}{1 + x^6}, & \overline{\widehat{f^{(1)}}} \star \overline{g^{(1)}} &= \frac{x + x^3 + x^4}{1 + x^6}, \end{aligned}$$

and

$$(26) \quad f^{(2)} = \frac{1}{(1+x)^2} + x \left( \frac{x + x^2 + x^4 + x^5}{1 + x^6} \right)^2 = \frac{1 + x^2 + x^3 + x^4 + x^5}{1 + x^6}.$$

In the same way one can compute  $\overline{f^{(2)}}$  and  $\overline{g^{(2)}}$ :

$$(27) \quad \overline{f^{(2)}} = \frac{x}{1 + x^3}, \quad \overline{g^{(2)}} = \frac{x + x^3}{1 + x^6}.$$

**THEOREM 2.2.** — *For any  $p \geq 0$ , the sequences (modulo 2)*

$$\{|\mathcal{E}_n^p|\}_{n \geq 0}, \quad \{|\overline{\mathcal{E}_n^p}|\}_{n \geq 0}, \quad \{|\Delta_n^p|\}_{n \geq 0}, \quad \{|\overline{\Delta_n^p}|\}_{n \geq 0}$$

*are all periodic. Furthermore,  $3 \cdot 2^k$  is a period if  $2^k + 1 \leq p \leq 2^{k+1}$ .*

*Proof.* — If  $p = 0, 1, 2$ , by equalities (22), (23), (26), and (27), these four sequences are periodic. Now, suppose  $p \geq 3$ . We shall prove by induction that  $3 \cdot 2^k$  is a period if  $2^k + 1 \leq p \leq 2^{k+1}$ . By (27), the conclusion is true for  $k = 1$ . Suppose that the conclusion is true for  $p \leq 2^k$ .

Consider now  $2^k + 1 \leq p \leq 2^{k+1}$ . If  $p = 2q$ , then  $2^{k-1} + 1 \leq q \leq 2^k$ , thus by Theorems 2.1.1 and 2.1.3, we have

$$(28) \quad \begin{aligned} |\mathcal{E}_{2n}^p| &\equiv |\mathcal{E}_n^q| \cdot |\Delta_n^q| + |\overline{\mathcal{E}_n^q}| \cdot |\overline{\Delta_n^q}|, \\ |\mathcal{E}_{2n+1}^p| &\equiv |\mathcal{E}_{n+1}^q| \cdot |\Delta_n^q| + |\overline{\mathcal{E}_{n+1}^q}| \cdot |\overline{\Delta_n^q}|. \end{aligned}$$

On the other hand, by the induction hypothesis, all sequences occurring on the right hand sides of the equalities (28) have period  $3 \cdot 2^{k-1}$ , and so do the product and the sum of these sequences. It follows that the sequences  $|\mathcal{E}_{2n}^p|$  and  $|\mathcal{E}_{2n+1}^p|$  are both  $3 \cdot 2^k$ -periodic and that this holds also for the sequence  $|\mathcal{E}_n^p|$ . The case  $p$  odd can be elucidated in the same way. Similar methods apply to the other three sequences.  $\square$

### 3. Automaticity properties.

In this section, we discuss further properties of the sequences introduced in Section 2. As the main result of this section, we prove that these sequences modulo 2 are all 2-automatic in the sense of Salon [12], [13]. As said in the introduction, automatic sequences have been widely and deeply studied in the recent years, as a general reference, one can read the survey by Dekking, Mendès France and van der Poorten [8] or the survey by Allouche [1]. For the two-dimensional automatic sequences, see [12], [13].

First of all, we recall one of the definitions of automatic sequences.

Let  $A$  be a finite alphabet and let  $A^*$  be the set of finite words. A substitution over  $A$  is a map  $\sigma: A \rightarrow A^*$ . If for any  $a \in A$ , the length of  $\sigma(a)$  (*i.e.*, the number of letters of the word  $\sigma(a)$ ) is equal to  $k$ , where  $k > 1$  is an integer, then  $\sigma$  is called a  $k$ -substitution. Furthermore, if  $a \in A$  is such that  $\sigma(a) = aw$ ,  $w \in A^*$ , then

$$\sigma^\omega(a) := \lim_{n \rightarrow \infty} \sigma^n(a)$$

defines an infinite word

$$x = x_1 x_2 \cdots x_n \cdots \in A^{\mathbb{N}}$$

(see [6], [4]), which is said to be generated by this  $k$ -substitution. Let  $B$  be another finite alphabet and let  $\tau$  be a map  $\tau: A \rightarrow B$ , then the sequence

$$\tau(x) = \{\tau(x_n)\}_{n \geq 1}$$

is also called a  $k$ -substitutive sequence (Cobham [6] proved that a  $k$ -substitutive sequence can be produced by a  $k$ -automaton and vice versa). We saw in Section 1 that the Thue-Morse sequence is generated by the 2-substitution  $\sigma: 1 \mapsto 10, 0 \mapsto 01$  and the period-doubling sequence by the substitution  $\theta: 1 \mapsto 10, 0 \mapsto 11$ .

The definition of a  $k$ -substitutive two-dimensional sequence [12], [13] is analogous, but a  $k$ -substitution in two dimensions associates with a single letter a “square” of letters of size  $k$ . For example define a two-dimensional 2-substitution as follows

$$0 \mapsto \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}, \quad 1 \mapsto \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}.$$

As previously, this operation can be iterated:

$$0 \mapsto \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \mapsto \begin{array}{cccc} & & 0 & 1 & 1 & 0 \\ & & 1 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 1 & 0 \end{array} \mapsto \dots$$

Let  $\{F_n^p\}_{n \geq 0, p \geq 0}$  be a double sequence. Its 2-kernel is the set of subsequences

$$\left\{ (F_{2^k n + i}^{2^k p + j})_{n \geq 0, p \geq 0}, 0 \leq k, 0 \leq i, j \leq 2^k - 1 \right\}.$$

It is known (see [12], [13]) that a sequence is 2-automatic if and only if its 2-kernel is finite.

**THEOREM 3.1.**

(i) *The sequences (modulo 2)*

$$\{|\mathcal{E}_n^p|\}_{n \geq 1, p \geq 0}, \quad \{|\overline{\mathcal{E}}_n^p|\}_{n \geq 1, p \geq 0}, \quad \{|\Delta_n^p|\}_{n \geq 1, p \geq 0}, \quad \{|\overline{\Delta}_n^p|\}_{n \geq 1, p \geq 0}$$

*are all 2-automatic.*

(ii) *For any  $n \geq 1$ , the sequences (modulo 2)*

$$\{|\mathcal{E}_n^p|\}_{p \geq 0}, \quad \{|\overline{\mathcal{E}}_n^p|\}_{p \geq 0}, \quad \{|\Delta_n^p|\}_{p \geq 0}, \quad \{|\overline{\Delta}_n^p|\}_{p \geq 0}$$

*are all 2-automatic.*

*Proof.* — For  $\alpha$  and  $\beta$  in  $\{0, 1\}$ , define two operations

$$S_\alpha^\beta u = \{u_{n+\alpha}^{p+\beta}\}_{n \geq 0, p \geq 0}, \quad D_\alpha^\beta u = \{u_{2n+\alpha}^{2p+\beta}\}_{n \geq 0, p \geq 0}$$

on sequences  $\{u_n^p\}_{n \geq 0, p \geq 0}$ . One has

$$(29) \quad D_\alpha^\beta S_{\alpha'}^{\beta'} = \begin{cases} D_{\alpha+\alpha'}^{\beta+\beta'} & \text{if } \alpha + \alpha' \leq 1 \text{ and } \beta + \beta' \leq 1, \\ S_0^1 D_{\alpha+\alpha'}^0 & \text{if } \alpha + \alpha' \leq 1 \text{ and } \beta + \beta' = 2, \\ S_1^0 D_0^{\beta+\beta'} & \text{if } \alpha + \alpha' = 2 \text{ and } \beta + \beta' \leq 1, \\ S_1^1 D_0^0 & \text{if } \alpha + \alpha' = 2 \text{ and } \beta + \beta' = 2. \end{cases}$$

Let  $\mathcal{E}$ ,  $\bar{\mathcal{E}}$ ,  $\Delta$ , and  $\bar{\Delta}$  stand for the sequences  $\{\mathcal{E}_n^p\}_{n \geq 0, p \geq 0}$ ,  $\{\bar{\mathcal{E}}_n^p\}_{n \geq 0, p \geq 0}$ ,  $\{\Delta_n^p\}_{n \geq 0, p \geq 0}$ , and  $\{\bar{\Delta}_n^p\}_{n \geq 0, p \geq 0}$  modulo 2.

Theorem 2.1 together with Proposition 2.1 can be reformulated in the following way:

$$(30) \quad \left\{ \begin{array}{ll} D_0^0 \mathcal{E} \equiv \mathcal{E} \cdot \Delta + \bar{\mathcal{E}} \cdot \bar{\Delta}, & D_0^1 \mathcal{E} \equiv S_0^1 \mathcal{E} \cdot \Delta + S_0^1 \bar{\mathcal{E}} \cdot \Delta + S_0^1 \bar{\mathcal{E}} \cdot \bar{\Delta}, \\ D_0^0 \bar{\mathcal{E}} \equiv \bar{\mathcal{E}} \cdot \Delta, & D_0^1 \bar{\mathcal{E}} \equiv S_0^1 \bar{\mathcal{E}} \cdot \Delta, \\ D_0^0 \Delta \equiv \Delta, & D_0^1 \Delta \equiv \Delta \cdot S_0^1 \Delta + \Delta \cdot S_0^1 \bar{\Delta} + \bar{\Delta} \cdot S_0^1 \Delta, \\ D_0^0 \bar{\Delta} \equiv 0, & D_0^1 \bar{\Delta} \equiv \Delta \cdot S_0^1 \bar{\Delta} + \bar{\Delta} \cdot S_0^1 \Delta, \\ D_1^0 \mathcal{E} \equiv S_1^0 \mathcal{E} \cdot \Delta + S_1^0 \bar{\mathcal{E}} \cdot \bar{\Delta}, & D_1^1 \mathcal{E} \equiv S_1^0 \mathcal{E} \cdot S_0^1 \Delta + S_1^0 \bar{\mathcal{E}} \cdot S_0^1 \Delta \\ & \quad + S_1^0 \bar{\mathcal{E}} \cdot S_0^1 \bar{\Delta}, \\ D_1^0 \bar{\mathcal{E}} \equiv S_1^0 \bar{\mathcal{E}} \cdot \Delta, & D_1^1 \bar{\mathcal{E}} \equiv S_1^0 \bar{\mathcal{E}} \cdot S_0^1 \Delta, \\ D_1^0 \Delta \equiv S_1^0 \Delta, & D_1^1 \Delta \equiv S_1^0 \Delta \cdot S_0^1 \Delta + S_1^0 \Delta \cdot S_0^1 \bar{\Delta} \\ & \quad + S_1^0 \bar{\Delta} \cdot S_0^1 \Delta, \\ D_1^0 \bar{\Delta} \equiv S_1^0 \bar{\mathcal{E}}, & D_1^1 \bar{\Delta} \equiv S_1^0 \Delta \cdot S_0^1 \bar{\Delta} + S_1^0 \bar{\Delta} \cdot S_0^1 \Delta. \end{array} \right.$$

(i) Set  $\mathcal{X} = \{\mathcal{E}, \bar{\mathcal{E}}, \Delta, \bar{\Delta}\}$  and  $\mathcal{Y} = \{S_\alpha^\beta F \mid F \in \mathcal{X}, \alpha = 0, 1, \beta = 0, 1\}$ . It results from (29) and (30) that, for any  $F \in \mathcal{Y}$  and  $\alpha$  and  $\beta$  in  $\{0, 1\}$ ,  $D_\alpha^\beta F$  can be expressed as a polynomial with coefficients in  $\text{GF}_2$  of the elements of  $\mathcal{Y}$ . As the elements of the 2-kernel of a sequence are obtained by successive applications of operators  $D_\alpha^\beta$ , it follows that the 2-kernels of the sequences  $\mathcal{E}$ ,  $\bar{\mathcal{E}}$ ,  $\Delta$ , and  $\bar{\Delta}$  are included in the set of sequences that are polynomials in sequences from  $\mathcal{Y}$ .

But, there is only a finite number of polynomial functions on  $\text{GF}_2$  with sixteen variables. Therefore, these 2-kernels are finite. Then, it results

from [12], [13] that the sequences  $|\mathcal{E}_n^p|_{n \geq 1, p \geq 0}$ ,  $|\Delta_n^p|_{n \geq 1, p \geq 0}$ ,  $|\overline{\mathcal{E}}_n^p|_{n \geq 1, p \geq 0}$ , and  $|\overline{\Delta}_n^p|_{n \geq 1, p \geq 0}$  (modulo 2) are 2-automatic.

(ii) An immediate consequence of a result of Salon [12], [13] is that, if the double sequence  $(F_n^p)_{n,p}$  is 2-automatic, then, for any fixed  $n \geq 1$ , the sequence  $(F_n^p)_p$  is 2-automatic, which proves our claim.

Alternatively we give a direct proof. Let  $\epsilon = \epsilon_0 \epsilon_1 \cdots \epsilon_n \cdots \in \{0, 1\}^{\mathbb{N}}$  be the Thue-Morse sequence. For  $n \geq 1$ , let  $\Omega_{2n+1}$  be the set of all subwords of  $\epsilon$  of length  $2n + 1$ . Now the 2-substitution  $\sigma$  above induces a new 2-substitution  $\sigma_n$  on  $\Omega_{2n+1}$  in the following way: let  $\omega = \omega_0 \omega_1 \cdots \omega_{2n}$  be an element of  $\Omega_{2n+1}$ ; if

$$\sigma(\omega) = \sigma(\omega_0 \omega_1 \cdots \omega_{2n}) = \sigma(\omega_0) \cdots \sigma(\omega_{2n}) = \eta_0 \eta_1 \cdots \eta_{4n+1},$$

then we set

$$\sigma_n(\omega) = (\eta_0 \eta_1 \cdots \eta_{2n})(\eta_1 \eta_2 \cdots \eta_{2n+1}) \in \Omega_{2n+1}^2.$$

It is easy to check that

$$\sigma_n^\omega(u_0) = \lim_{k \rightarrow \infty} \sigma_n^k(u_0) = u_0 u_1 \cdots \in \Omega_{2n+1}^{\mathbb{N}},$$

where  $u_j$  is the block  $\epsilon_j \epsilon_{j+1} \cdots \epsilon_{j+2n}$ . This means that the sequence  $u$  is generated by the 2-substitution  $\sigma_n$ . Now define the map  $\tau_n : \Omega_{2n+1} \rightarrow \{0, 1\}$  by  $\tau_n(u_p) := |\mathcal{E}_n^p| \pmod{2}$ . Then, the image of the sequence  $(u_p)_{p \geq 0}$  under  $\tau_n$  is equal to the sequence  $\{|\mathcal{E}_n^p|\}_{p \geq 0}$  modulo 2. Hence the sequence  $\{|\mathcal{E}_n^p|\}_{p \geq 0}$  modulo 2 is 2-automatic. It can be proved in the same way that the other three sequences are 2-automatic.  $\square$

*Remark 3.1.* — If  $u = u_0 u_1 \cdots u_n \cdots$  is an automatic sequence that can be generated by a primitive substitution, then the sequence is minimal: any factor of the sequence  $u$  occurs in  $u$  with a non-zero frequency [6]. Hence by Theorem 3.1, we have the following corollary.

**COROLLARY 3.1.** — *For any  $p \geq 0$  (resp.  $n \geq 1$ ), there are infinitely many numbers  $n$  (resp.  $p$ ), such that  $|\mathcal{E}_n^p| \neq 0$ . The same property holds also for the sequence  $|\Delta_n^p|$ .*

*Proof.* — For a fixed  $n \geq 1$  consider the map  $\Lambda : \{0, 1\}^{2n-1} \rightarrow \{0, 1\}$  defined by

$$\Lambda(a_1, \dots, a_{2n-1}) = \det\{a_{i+j-1}\}_{1 \leq i, j \leq n}.$$

One has  $\Lambda(\epsilon_p, \dots, \epsilon_{p+2n-2}) = |\mathcal{E}_n^p|$ , for all  $p \geq 0$ . As the Thue-Morse sequence is minimal, each block  $a_1 \cdots a_{2n-1}$  that occurs in the Thue-Morse sequence occurs an infinite number of times and with bounded gaps. Hence we have the same property for the “block” (actually the letter)  $\Lambda(a_1, \dots, a_{2n-1})$  in the sequence  $(|\mathcal{E}_n^p|)_{p \geq 0}$ . Hence the frequency of any letter in the sequence  $(|\mathcal{E}_n^p|)_{p \geq 0}$  is strictly positive and this sequence is not 0 identically.

For a fixed  $p \geq 0$  the sequence  $n \mapsto (|\mathcal{E}_n^p|)_n$  is periodic and it is not identical to zero, (one has  $|\mathcal{E}_1^p| = \epsilon_p$ ). Hence any letter occurs an infinite number of times in the sequence  $(|\mathcal{E}_n^p|)_{n \geq 1}$ .  $\square$

#### 4. Applications.

Now we consider another form of the Thue-Morse sequence which will be used in the applications below. In the definition of the introduction, if we take  $a = 1$ ,  $b = -1$ , then we obtain an infinite sequence  $\eta = \eta_0 \eta_1 \cdots \eta_n \cdots \in \{1, -1\}^{\mathbb{N}}$  which satisfies the recurrence equations  $\eta_0 = 1$ ,  $\eta_{2n} = \eta_n$ ,  $\eta_{2n+1} = -\eta_n$ . We define  $\pi_n = -\eta_{n+1} \eta_n$  (note that  $\{\frac{1}{2}(1 + \pi_n)\}$  is nothing but the period doubling sequence). The Hankel determinants corresponding to  $\eta$  and  $\pi$  are denoted by  $|A_n^p|$  and  $|B_n^p|$ .

We clearly have

$$(31) \quad \eta_n = 2\epsilon_n - 1, \quad \pi_n = 2|\delta_n| - 1.$$

The following proposition relates  $|A_n^p|$  and  $|B_n^p|$  to  $|\mathcal{E}_n^p|$  and  $|\Delta_n^p|$ .

PROPOSITION 4.1. — *We have*

$$\begin{aligned} 2^{1-n} |A_n^p| &= |\overline{\mathcal{E}_n^p}| + 2|\mathcal{E}_n^p| \equiv |\overline{\mathcal{E}_n^p}|, \\ 2^{1-n} |B_n^p| &= |\widetilde{\Delta}_n^p| + 2|\widetilde{\Delta}_n^p| \equiv |\overline{\Delta}_n^p|, \end{aligned}$$

where  $\widetilde{\Delta}_n^p$  is the  $(n, p)$ -Hankel matrix of the sequence  $|\delta_j|$ .

*Proof.* — This proposition results from Lemma 1.4 and Formula (31).  $\square$

A direct consequence of Theorems 2.1, 2.2, 3.1 and Proposition 4.1 is the following.

THEOREM 4.1. — *The sequences modulo 2*

$$\begin{aligned} \{2^{1-n}|A_n^p|\}_{n \geq 1, p \geq 0}, & \quad \{2^{1-n}|\overline{A_n^p}|\}_{n \geq 1, p \geq 0}, \\ \{2^{1-n}|B_n^p|\}_{n \geq 1, p \geq 0}, & \quad \{2^{1-n}|\overline{B_n^p}|\}_{n \geq 1, p \geq 0} \end{aligned}$$

are 2-automatic.

For any  $n \geq 1$ , the sequences modulo 2

$$\begin{aligned} \{2^{1-n}|A_n^p|\}_{p \geq 0}, & \quad \{2^{1-n}|\overline{A_n^p}|\}_{p \geq 0}, \\ \{2^{1-n}|B_n^p|\}_{p \geq 0}, & \quad \{2^{1-n}|\overline{B_n^p}|\}_{p \geq 0} \end{aligned}$$

are 2-automatic.

For any  $p \geq 0$ , the sequences modulo 2

$$\begin{aligned} \{2^{1-n}|A_n^p|\}_{n \geq 1}, & \quad \{2^{1-n}|\overline{A_n^p}|\}_{n \geq 1}, \\ \{2^{1-n}|B_n^p|\}_{n \geq 1}, & \quad \{2^{1-n}|\overline{B_n^p}|\}_{n \geq 1} \end{aligned}$$

are periodic. □

From the theorem above and Propositions 2.2, 4.1, we get the following corollary.

COROLLARY 4.1. — *For  $n \geq 1$ , one has  $2^{1-n}|A_n^0| \equiv 1 \pmod{2}$ . In particular,  $|A_n^0| \neq 0$  for  $n \geq 1$ . □*

By a remark similar to Remark 3.1, we have the following result.

COROLLARY 4.2. — *For any  $n \geq 1$  (resp.  $p \geq 0$ ) there are infinitely many integers  $p$  (resp.  $n$ ) such that  $|A_n^p| \neq 0$ . The conclusion is also valid for the sequence  $(|B_n^p|)$ . □*

Now we discuss the strongly nonrepetitive structure of the Thue-Morse sequence.

Let  $A = \{a, b\}$ . If  $w \in A^*$  is a finite word, we denote by  $\tilde{w}$  the word obtained by flipping  $a$  and  $b$  in it.

Let  $u = u_0u_1 \dots u_n \dots \in A^{\mathbb{N}}$  and let  $w_{p,n} = u_p \dots u_{p+n}$  be a factor (american terminology: subword) of  $u$ . We say that a word  $w_{p,2n}$  is

nonrepetitive (resp. strongly nonrepetitive) if, for all  $k$  and  $\ell$  such that  $p \leq k < \ell \leq p + n$ , we have  $w_{k,n} \neq w_{\ell,n}$  (resp.  $w_{k,n} \neq w_{\ell,n}$  and  $w_{k,n} \neq \tilde{w}_{\ell,n}$ ). It is known that any factor  $w_{p,2n}$  of the Thue-Morse sequence is nonrepetitive [14], [15]. Notice now that if  $w_{k,n} = \eta_k \eta_{k+1} \cdots \eta_{k+n}$ , then  $\tilde{w}_{k,n}$  is just the word  $(-\eta_k)(-\eta_{k+1}) \cdots (-\eta_{k+n})$ . Hence, if the  $(p, n)$ -order Hankel determinant  $|A_n^p|$  is nonzero, then the word  $w_{p,2n}$  is strongly nonrepetitive. Thus, by Corollaries 4.1 and 4.2, we obtain the following theorem.

**THEOREM 4.2.** — *Let  $u = u_0 u_1 \dots u_n \dots \in A^{\mathbb{N}}$  be the Thue-Morse sequence, then, for any  $n \geq 1$ , the words  $w_{0,2n}$  are strongly nonrepetitive. Furthermore, for any  $p \geq 1$ , there are infinitely many  $n$  such that the words  $w_{p,2n}$  are strongly nonrepetitive.*  $\square$

*Remark 4.1.* — Notice that  $w_{2,2} = bab$ ,  $w_{2,1} = ba$ ,  $w_{3,1} = ab$ , so  $w_{2,1} = \tilde{w}_{3,1}$ , that is, the word  $w_{2,2}$  is not strongly nonrepetitive. Thus, in this sense, the theorem cannot be improved.

*Remark 4.2.*

(i) Let  $v = v_0 v_1 \cdots v_n \cdots \in A^{\mathbb{N}}$  be the period-doubling sequence. For  $p = 0$ , we do not have the same result as for the Thue-Morse sequence. In fact, consider  $w_{0,2} = aba$ , then  $w_{0,1} = \tilde{w}_{1,2} = ab$ . Nevertheless, by Corollary 4.2, we still have that, for any  $p$ , there are infinitely many  $n$ , such that the words  $w_{p,2n}$  are strongly nonrepetitive.

(ii) From Proposition 2.2,  $|\Delta_n^0| \neq 0$ , hence for any  $n \geq 1$ , the subword  $w_{0,2n}$  of  $w$  is nonrepetitive. Furthermore, this result cannot be improved, for example,  $w_{4,4} = ababa$ , but  $w_{4,2} = w_{6,2} = aba$ .

On the other hand, from Remark 1.1, we see that, nonrepetitivity for the period-doubling sequence is equivalent to strong nonrepetitivity for the Thue-Morse sequence, hence, the first conclusion of the above theorem can be derived from Remark 4.2 (ii).

Now consider again the Thue-Morse sequence  $\eta = \eta_0 \eta_1 \cdots \eta_n \cdots$  in  $\{1, -1\}^{\mathbb{N}}$ . Set

$$f(x) = \sum_{n \geq 0} \eta_n x^n;$$

then  $f(x)$  is a transcendental function, and Cobham [5] proved that  $f(x)$  is

the unique solution of the following functional equation:

$$F(x^2) = \frac{F(x)}{1+x} - \frac{x}{(1+x)(1+x^2)}, \quad |x| < 1,$$

such that  $F(0) = 1$ , see also [7], [8].

Thus, we are naturally led to consider approximating  $f(x)$  by rational functions. Nice candidates are the Padé approximants (if they exist).

A  $(p, q)$ -order Padé approximant of  $f$ , noted  $\left[\frac{p}{q}\right]_f$ , is a rational function  $P(x)/Q(x)$  whose denominator has degree  $q$  and whose numerator has degree  $p$  such that

$$f(x) - \frac{P(x)}{Q(x)} = O(x^{p+q+1}), \quad x \rightarrow 0.$$

In particular, the approximants  $\left[\frac{k-1}{k}\right]_f$  for  $k \geq 1$  play an important rôle in the study of Padé approximant theory (for a general reference, see [2]). By a classical result, if  $|A_n| \neq 0$ , then the Padé approximant  $\left[\frac{n-1}{n}\right]_f$  exists (furthermore, it can be expressed explicitly, see [3], pp. 34–36). Hence by Corollary 4.1, we have the following theorem.

**THEOREM 4.3.** — *Let  $f(x) = \sum_{n \geq 0} \eta_n x^n$ ; then for any  $n \geq 1$ , the  $(n-1, n)$ -order Padé approximant of  $f$  exists.*  $\square$

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